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A FRESNEL TYPE CLASS ON FUNCTION SPACE

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ABSTRACT. In this paper we define a Banach algebra on very general function space induced by a generalized Brownian motion process rather than on Wiener space, but the Banach algebra can be considered as a generalization of Fresnel class defined on Wiener space. We then show that several interesting functions in quantum mechanic are elements of the class.

1. INTRODUCTION

Abstract Wiener spaces have been of interest since the work of Gross [6] and are currently being used as a framework in the study of the Fresnel and Feynman integrals. The Feynman integral arose in nonrelativistic quantum mechanics and has been studied by mathematicians and theoretical physicists. The Fresnel integral has been defined in Hilbert space [1], classical Wiener space [2] and abstract Wiener space [9] settings and used as an approach to the Feynman integral.

Let (H, B, i) be an abstract Wiener space. The Fresnel class $\mathcal{F}(B)$ of B is the class of all stochastic Fourier transforms of complex Borel measures on $\mathcal{B}(H)$, the Borel class of H. There are several results insuring that various functions of interest in connection with the Feynman integral and quantum mechanics are in $\mathcal{F}(B)$, for example see [1, 2, 8, 9, 10]. It is well known that the Fresnel class $\mathcal{F}(B)$ is a Banach algebra.

In this paper we define a Banach algebra on very general function space $C_{a,b}[0,T]$ rather than on abstract Wiener space, but the Banach algebra can be considered as a generalization of Fresnel class defined on abstract Wiener space. The function space $C_{a,b}[0,T]$ induced by a generalized Brownian motion was introduced by J. Yeh in [12] and was used extensively by Chang, Chung and Skoug [3, 5]. We then show that

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several interesting functions in quantum mechanic are elements of the class. Recall that the Wiener process is free of drift and is stationary in time, while the stochastic process considered in this paper is subject to a drift $a(t)$ and is nonstationary in time.

2. DEFINITIONS AND PRELIMINARIES

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with the density function

$$K(\vec{t},\vec{\eta}) = \left((2\pi)^n \prod_{j=1}^{n} \left(b(t_j) - b(t_{j-1})\right)\right)^{-1/2} \times \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\right\}$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [13, p. 18-20] Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [13, p. 187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

We shall say that two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-almost everywhere(s-a.e.) if for each $\rho > 0$, $F(\rho x) = G(\rho x)$ for almost all $x \in C_{a,b}[0,T]$. We denote this equivalence relation by $F \approx G$.

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\}$$

where $|a|(t)$ denotes the total variation of the function $a(\cdot)$ on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ almost everywhere on $[0,T]$. Furthermore, $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$. Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (v,\phi_j)_{a,b}\phi_j(t)dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$ and that if v is of bounded variation on $[0,T]$, then the PWZ stochastic integral $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_{a,b}[0,T]$. For more details, see [5].

Remark 2.1. For each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ is a Gaussian random variable on $C_{a,b}[0,T]$ with mean $\int_0^T v(s)da(s)$ and variance $\int_0^T v^2(s)db(s)$. Note that for all $u, v \in L^2_{a,b}[0,T]$,

$$\int_{C_{a,b}[0,T]} \langle u, x\rangle\langle v, x\rangle d\mu(x) = \int_0^T u(s)v(s)db(s) + \int_0^T u(s)da(s)\int_0^T v(s)da(s).$$

Hence we see that for all $u, v \in L^2_{a,b}[0,T]$, $\int_0^T v(s)u(s)db(s) = 0$ if and only if $\langle u, x\rangle$ and $\langle v, x\rangle$ are independent random variables.

3. THE FRESNEL TYPE CLASS ON FUNCTION SPACE

In this section we define a Banach algebra $\mathcal{F}(C_{a,b}[0,T])$ of functionals defined on function space $C_{a,b}[0,T]$. First, we will introduce a separable Hilbert space. Let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D_t : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$D_t w = z(t) = \frac{w'(t)}{b'(t)}. \tag{3.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T D_t w_1 D_t w_2 db(t) = \int_0^T z_1(t) z_2(t) db(t)$$

is a separable Hilbert space. Furthermore, $(C'_{a,b}[0,T], C_{a,b}[0,T], \mu)$ is an abstract Wiener space. For more details, see [11].

Note that the linear operator given by equation (3.1) is an isomorphism. In fact, the inverse operator $D_t^{-1} : L^2_{a,b}[0,T] \to C'_{a,b}[0,T]$ is given by the formula

$$D_t^{-1} z = \int_0^t z(s)db(s)$$

and D_t^{-1} is a bounded operator since

$$\begin{aligned} \|D_t^{-1} z\|_{C'_{a,b}} &= \left\| \int_0^t z(s)db(s) \right\|_{C'_{a,b}} = \left(\int_0^T z^2(s)db(s) \right)^{1/2} \\ &\leq \left(\int_0^T z^2(s) d[b(s) + |a|(s)] \right)^{1/2} = \|z\|_{a,b}. \end{aligned}$$

Thus by the open mapping theorem, we see that D_t is also bounded and there exist positive real numbers α and β such that $\alpha\|w\|_{C'_{a,b}} \leq \|D_t w\|_{a,b} \leq \beta\|w\|_{C'_{a,b}}$ for all $w \in C'_{a,b}[0,T]$. Thus we see that the Borel σ-algebra on $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$ is given by

$$\mathcal{B}(C'_{a,b}[0,T]) = \{D_t^{-1}(E) : E \in \mathcal{B}(L^2_{a,b}[0,T])\}.$$

Throughout this paper, for $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, we will use the notation $(w, x)^\sim$ instead of $\langle z, x \rangle \equiv \langle D_t w, x \rangle$. Then we have the following assertions.

(1) For each $w \in C'_{a,b}[0,T]$, the random variable $x \mapsto (w,x)^{\sim}$ is Gaussian with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$.

(2) $(w,\alpha x)^{\sim} = (\alpha w, x)^{\sim} = \alpha(w,x)^{\sim}$ for any real number α, $w \in C'_{a,b}[0,T]$ and $x \in C'_{a,b}[0,T]$.

(3) If $\{w_1, w_2, \cdots, w_n\}$ is an orthonormal set in $C'_{a,b}[0,T]$, then the random variables $(w_i,x)^{\sim}$'s are independent.

Next, we define a class of functionals on $C_{a,b}[0,T]$ like a Fresnel class of an abstract Wiener space.

Definition 3.1. Let $\mathcal{M}(C'_{a,b}[0,T])$ be the space of complex-valued, countably additive (and hence finite) Boreal measures on $C'_{a,b}[0,T]$. The Banach algebra

$$\mathcal{F}(C_{a,b}[0,T])$$

consists of those functionals F on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\} df(w)$$

for s-a.e. $x \in C_{a,b}[0,T]$ where the associated measure f is an element of

$$\mathcal{M}(C'_{a,b}[0,T]).$$

We call $\mathcal{F}(C_{a,b}[0,T])$ the Fresnel type class of the function space $C_{a,b}[0,T]$.

Remark 3.2. (1) $\mathcal{M}(C'_{a,b}[0,T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(2) One can show that the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0,T]} |df(w)|.$$

(3) In [5] Chang and Skoug introduced a Banach algebra $S(L^2_{a,b}[0,T])$ of functionals on $C_{a,b}[0,T]$ given by

$$S(L^2_{a,b}[0,T]) = \left\{ F : F(x) \approx \int_{L^2_{a,b}[0,T]} \exp\{i\langle v,x\rangle\} d\sigma(v),\ \sigma \in \mathcal{M}(L^2_{a,b}[0,T]) \right\},$$

and then showed that generalized analytic Feynman integrals and generalized analytic Fourier-Feynman transforms of functionals in $S(L^2_{a,b}[0,T])$ exist under appropriate conditions. If

$$F(x) \approx \int_{L^2_{a,b}[0,T]} \exp\{i\langle v,x\rangle\} d\sigma(v)$$

for some $\sigma \in \mathcal{M}(L^2_{a,b}[0,T])$, then we have

$$F(x) \approx \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim\} d(\sigma \circ D_t)(w)$$

where D_t is given by equation (3.1) above. Conversely, if

$$F(x) \approx \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim\} df(w)$$

for some $f \in \mathcal{M}(C'_{a,b}[0,T])$, then we have

$$F(x) \approx \int_{L^2_{a,b}[0,T]} \exp\{i\langle v,x\rangle\} d(f \circ D_t^{-1})(v).$$

Thus we have that $F \in S(L^2_{a,b}[0,T])$ if and only if $F \in \mathcal{F}(C_{a,b}[0,T])$.

4. FUNCTIONALS IN $\mathcal{F}(C_{a,b}[0,T])$

In this section we will prove that several interesting functions in quantum mechanic are elements of the Fresnel type class. We begin by stating an unsymmetric Fubini theorem and a lemma from [7, 8] which we use to prove our theorems. We will denote $\mathcal{M}(X)$ the set of all complex finite Borel measures on $\mathcal{B}(X)$ for a space X.

Theorem 4.1. *Let $(Y, \mathcal{Y}, \gamma)$ be a σ-finite measure space and let $(Z, \mathcal{Z})$ be a measurable space. For γ-a.e. $y \in \mathcal{Y}$, let σ_y be a complex-valued, countably additive measure on $(Z, \mathcal{Z})$ of finite total variation $\|\sigma_y\|$. Suppose that, for any B in $\mathcal{Z}$, $\sigma_y(B)$ is a $\mathcal{Y}$-measurable function of y. Then*

1. *for any E in σ-algebra $\mathcal{Y} \times \mathcal{Z}$, $\sigma_y(E^{(y)})$ is a $\mathcal{Y}$-measurable function of y, and*
2. *for any bounded, complex-valued, $\mathcal{Y} \times \mathcal{Z}$ measurable function $\phi(y,z)$ on $Y \times Z$, $\int_Z \phi(y,z) d\sigma_y(z)$ is a $\mathcal{Y}$-measurable function of y.*

If we add assumption that $\|\sigma_y\| \leq h(y)$, where h is in $L^1(Y, \mathcal{Y}, \gamma)$, and define ν on $\mathcal{Y} \times \mathcal{Z}$ by

$$\nu(E) = \int_Y \sigma_y(E^{(y)}) d\gamma(y),$$

then

3. *ν is a complex-valued, countably additive measure on $\mathcal{Y} \times \mathcal{Z}$ with $\|\nu\| \leq \|h\|_1$, and*

(4) *If* $\phi(y,z)$ *is bounded and* $\mathcal{Y}\times\mathcal{Z}$*-measurable, then* $\int_Z \phi(y,z)d\sigma_y(z)$ *is in* $L^1(Y,\mathcal{Y},\gamma)$, *and we have*

$$\int_Y \left[\int_Z \phi(y,z)d\sigma_y(z)\right]d\gamma(y) = \int_{Y\times Z} \phi(y,z)d\nu(y,z). \tag{4.1}$$

Lemma 4.2 ([8, Corollary 3.1]). *Let* $\{\sigma_s : 0 \le s \le T\}$ *be a family from* $\mathcal{M}(\mathbb{R})$ *such that* $\sigma_s(E)$ *is a Borel measurable function of* s *for every* E *in* $\mathcal{B}(\mathbb{R})$. *Then* $\|\sigma_s\|$ *is a Borel measurable function of* s.

Next, we give a definition of potential functions which is used in this section.

Definition 4.3. Let $\mathcal{G}$ be the set of all complex-valued functions on $[0,T]\times\mathbb{R}$ of the form

$$\theta(s,u) = \int_{\mathbb{R}} \exp\{iuv\}d\sigma_s(v) \tag{4.2}$$

where $\{\sigma_s : 0 \le s \le T\}$ is a family from $\mathcal{M}(\mathbb{R})$ satisfying the following two conditions:

(1) For every $E \in \mathcal{B}(\mathbb{R})$, $\sigma_s(E)$ is Borel measurable in s,

(2) $\||\sigma_s\||_{0,b} = \int_0^T \|\sigma_s\|db(s) < +\infty$.

Remark 4.4. In [8], θ was a complex-valued function on $[0,T]\times\mathbb{R}^n$ given by

$$\theta(s,\vec{u}) = \int_{\mathbb{R}^n} \exp\{i(\vec{u},\vec{v})\}d\sigma_s(\vec{v})$$

where $\{\sigma_s : 0 \le s \le T\}$ is a family from $\mathcal{M}(\mathbb{R}^n)$ satisfying

(1) for every $E \in \mathcal{B}(\mathbb{R}^n)$, $\sigma_s(E)$ is Borel measurable in s,

(2) $\|\sigma_s\| \in L^1[0,T]$, i.e., $\||\sigma_s\||_1 = \int_0^T \|\sigma_s\|ds < +\infty$.

In Section 2, we assumed that $b(\cdot)$ is a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$. From this, we see that $\|\cdot\|_1$ and $\|\cdot\|_{0,b}$ are equivalent norms on $L^1[0,T]$.

We also need a lemma for our results.

Lemma 4.5 ([8, Corollary 3.2]). *Let* $\theta : [0,T]\times\mathbb{R}\to\mathbb{C}$ *be given by* (4.2) *where* $\{\sigma_s : 0 \le s \le T\}$ *is a family from* $\mathcal{M}(\mathbb{R})$ *satisfying the condition* (1) *in Definition* 4.3. *Then* θ *is Borel measurable.*

Theorem 4.6. *Let* $t \in [0,T]$ *be fixed. Let* $\theta(t,\cdot) : \mathbb{R}\to\mathbb{C}$ *be given by*

$$\theta(t,u)=\int_{\mathbb{R}}\exp\{iuv\}d\sigma_t(v)$$

where σ_t is in $\mathcal{M}(\mathbb{R})$. Then $G_t : C_{a,b}[0,T]\to\mathbb{C}$ given by

$$G_t(x)=\theta(t,x(t))$$

is in the Banach algebra $\mathcal{F}(C_{a,b}[0,T])$.

Proof. Applying Lemma 4.5, we see that G_t is $\mathcal{B}(C_{a,b}[0,T])$-measurable. We seek a measure f_t in $\mathcal{M}(C'_{a,b}[0,T])$ such that for s-a.e. $x\in C_{a,b}[0,T]$

$$G_t(x)=\int_{C'_{a,b}[0,T]}\exp\{i(w,x)^\sim\}df_t(w).$$

For given $t\in[0,T]$, let $\Theta_t:\mathbb{R}\to C'_{a,b}[0,T]$ be defined by

$$\Theta_t(u)(s)=\int_0^s u\chi_{[0,t)}(\tau)db(\tau). \tag{4.3}$$

Then Θ_t is a measurable function of u and $D_s\Theta_t(u)=u\chi_{[0,t)}(s)$. We claim that $f_t=\sigma_t\circ\Theta_t^{-1}$ is the desired measure. Let $\rho>0$ be given. We need to show that for μ-a.e. $x\in C_{a,b}[0,T]$

$$G_t(\rho x)=\theta(t,\rho x(t))=\int_{C'_{a,b}[0,T]}\exp\{i(w,\rho x)^\sim\}df_t(w).$$

But

$$\begin{aligned}G_t(\rho x)&=\theta(t,\rho x(t))\\&=\int_{\mathbb{R}}\exp\{i\rho ux(t)\}d\sigma_t(u)\\&=\int_{\mathbb{R}}\exp\left\{i\rho\int_0^t udx(s)\right\}d\sigma_t(u)\\&=\int_{\mathbb{R}}\exp\left\{i\rho\int_0^T u\chi_{[0,t)}(s)dx(s)\right\}d\sigma_t(u)\\&=\int_{\mathbb{R}}\exp\left\{i\rho\int_0^T D_s\Theta_t(u)dx(s)\right\}d\sigma_t(u)\\&=\int_{\mathbb{R}}\exp\{i\rho(\Theta_t(u),x)^\sim\}d\sigma_t(u).\end{aligned}$$

Now, using the change of variables theorem, this last expression equals

$$\int_{C'_{a,b}[0,T]}\exp\{i\rho(w,x)^\sim\}d(\sigma_t\circ\Theta_t^{-1})(w)$$

as desired. □

Theorem 4.7. *Let θ be in $\mathcal{G}$ and be given by (4.2). Let $G : C_{a,b}[0,T] \to \mathbb{C}$ be given by*

$$G(x) = \int_0^T G_t(x)dt = \int_0^T \theta(t, x(t))dt.$$

Then G is in $\mathcal{F}(C_{a,b}[0,T])$.

Proof. We will use Theorem 4.1 and Lemma 4.2 with

$$(Y, \mathcal{Y}, \gamma) = ([0,T], \mathcal{B}([0,T]), m_L)$$

and $(Z, \mathcal{Z}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where m_L is the Lebesgue measure. By Lemma 4.2, $\|\sigma_t\|$ is a $\mathcal{B}([0,T])$-measurable function of t, and by (1) of Theorem 4.1, for each Borel subset E of $[0,T] \times \mathbb{R}$, $\sigma_t(E^{(t)})$ is a $\mathcal{B}([0,T])$-measurable function of t. Define a set function ν by

$$\nu(E) = \int_0^T \sigma_t(E^{(t)})dt \tag{4.4}$$

for each $E \in \mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{R})$. Then, by (3) of Theorem 4.1, ν is a complex Borel measure on $\mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{R})$ and

$$\|\nu\| \le \int_0^T \|\sigma_t\|dt \le \left(\sup_{t\in[0,T]} \frac{1}{b'(t)} \right) \int_0^T \|\sigma_t\|db(t) < +\infty.$$

By (2) of Theorem 4.1, for any bounded Borel measurable function ϕ on $[0,T] \times \mathbb{R}$ $\int_{\mathbb{R}} \phi(t,v)d\sigma_t(v)$ is a $\mathcal{B}([0,T])$-measurable function of t and, by equation (4.1)

$$\int_0^T \left[\int_{\mathbb{R}} \phi(t,v)d\sigma_t(v) \right] dt = \int_{[0,T]\times\mathbb{R}} \phi(t,v)d\nu(t,v).$$

For each $x \in C_{a,b}[0,T]$, $\phi(t,v) = \exp\{ivx(t)\}$ is a bounded Borel measurable function in $(t,v) \in [0,T] \times \mathbb{R}$. Hence

$$\begin{aligned} G(x) &= \int_0^T \theta(t, x(t))dt \\ &= \int_0^T \left[\int_{\mathbb{R}} \exp\{ivx(t)\}d\sigma_t(v) \right] dt \\ &= \int_{[0,T]\times\mathbb{R}} \exp\{ivx(t)\}d\nu(t,v). \end{aligned} \tag{4.5}$$

Let $\Phi : [0,T] \times \mathbb{R} \to C'_{a,b}[0,T]$ be a map defined by

$$\Phi(t,v) = \Theta_t(v) \tag{4.6}$$

where Θ_t is given by equation (4.3). Using equations (4.5), (4.3), (4.6) and the change of variables theorem, we have that for all $\rho > 0$

$$\begin{aligned} G(\rho x) &= \int_{[0,T]\times\mathbb{R}} \exp\{i\rho v x(t)\} d\nu(t,v) \\ &= \int_{[0,T]\times\mathbb{R}} \exp\{i\rho(\Theta_t(v), x)^\sim\} d\nu(t,v) \\ &= \int_{[0,T]\times\mathbb{R}} \exp\{i\rho(\Phi(t,v), x)^\sim\} d\nu(t,v) \\ &= \int_{C'_{a,b}[0,T]} \exp\{i\rho(w,x)^\sim\} d(\nu\circ\Phi^{-1})(w). \end{aligned}$$

□

Lemma 4.8. *For $m = 1, 2, \cdots$, let $F_m \in \mathcal{F}(C_{a,b}[0,T])$ and let*

$$\sum_{m=1}^{\infty} \|F_m\| < +\infty. \tag{4.7}$$

Then F is in $\mathcal{F}(C_{a,b}[0,T])$ where $F(x) = \sum_{m=1}^{\infty} F_m(x)$ for s-a.e. $x \in C_{a,b}[0,T]$.

Proof. Let f_m be the associated measure of F_m for each $m \in \mathbb{N}$. We define a measure f on $\mathcal{B}(C'_{a,b}[0,T])$ as follows: if $E \in \mathcal{B}(C'_{a,b}[0,T])$

$$f(E) = \sum_{m=1}^{\infty} f_m(E). \tag{4.8}$$

The series (4.8) above converges absolutely since it follows from (4.7) that

$$\sum_{m=1}^{\infty} \|f_m\| = \sum_{m=1}^{\infty} \|F_m\| < +\infty. \tag{4.9}$$

Clearly, f is a measure. Now by equations (4.8) and (4.9), we have that

$$\begin{aligned} F(x) = \sum_{m=1}^{\infty} F_m(x) &= \sum_{m=1}^{\infty} \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim\} df_m(w) \\ &= \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim\} df(w) \end{aligned}$$

for s-a.e. $x \in C_{a,b}[0,T]$, and $F \in \mathcal{F}(C_{a,b}[0,T])$. □

Our next theorem comes basically from the fact that $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra. This theorem is relevant to quantum mechanics where exponential functions play a prominent role.

Theorem 4.9. *Let θ and G be as in Theorem 4.7 above. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be given by*

$$F(x) = \exp\{G(x)\} = \exp\left\{\int_0^T \theta(t, x(t))dt\right\}$$

Then F is in $\mathcal{F}(C_{a,b}[0,T])$.

Proof. For each $m \in \mathbb{N}$, let $F_m(x) = (1/m!)[G(x)]^m$. Then $F(x) = 1+\sum_{m=1}^{\infty} F_m(x)$. We know from the proof of Theorem 4.7 that G is associated with the measure $g = \nu \circ \Phi^{-1}$ where ν is given by (4.4) and Φ is given by (4.6). Because convolution (of measures) is taken over into pointwise multiplication by the map from $\mathcal{M}(C'_{a,b}[0,T])$ onto $\mathcal{F}(C_{a,b}[0,T])$, the measure $f_m \equiv (1/m!)g * \cdots * g$ (m convolutions) is associated with $F_m(x) = (1/m!)[G(x)]^m$. Now

$$\|F_m\| = \|f_m\| = \|(1/m!)g * \cdots * g\| \leq \frac{1}{m!}\|g\|^m$$

and so, of course, $\sum_{m=1}^{\infty} \|F_m\| < +\infty$. Hence it follows that F is in $\mathcal{F}(C_{a,b}[0,T])$ by Lemma 4.8. □

5. Remark and Example

In [3], Chang and Chung showed that the function U on $[0,T] \times \mathbb{R} \times \mathbb{R}$ defined by

$$U(t;\xi,\eta) = E[F_t|X_t = \eta](2\pi b(t))^{-1/2} \exp\left\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\right\} \tag{5.1}$$

satisfies the generalized Kac-Feynman integral equation

$$\begin{aligned} U(t;\xi,\eta) = {} & (2\pi b(t))^{-1/2} \exp\left\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\right\} \\ & + \int_0^t \int_{\mathbb{R}} \vartheta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t) - b(s)))^{-1/2} \\ & \times \exp\left\{-\frac{((\zeta - a(s)) - (\eta - a(t)))^2}{2(b(t) - b(s))}\right\}d\zeta ds. \end{aligned} \tag{5.2}$$

In equation (5.1), F_t and X_t are $\mathcal{B}^D$-measurable functions on $\mathbb{R}^D$ defined by

$$F_t(x) = \exp\left\{\int_0^t \vartheta(s, x(s) + \xi)ds\right\} \text{ and } X_t(x) = x(t) + \xi$$

where $\vartheta(\cdot,\cdot)$ is a complex-valued Borel measurable function on $[0,T]\times\mathbb{R}$ for which F_t is μ-integrable for each $(t,\xi) \in [0,T]\times\mathbb{R}$. Actually, it is hold for the case that F_t and X_t are defined on the function space $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$. As a special case, by Theorem 4.9, we can see that for F_t with ϑ replaced with θ given by equation (4.2), $U(t;\xi,\eta)$ is a solution of the integral equation (5.2).

The integral equation (5.2) is equivalent to a partial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}b'(t)\frac{\partial^2 U}{\partial \eta^2} - a'(t)\frac{\partial U}{\partial \eta} + \vartheta(t,\eta)U(t;\xi,\eta), \tag{5.3}$$

which is the generalized form of the heat equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial \eta^2} + \vartheta U.$$

In other words, (5.1) is a solution of the partial differential equation (5.3). For more details, see [3] and [4].

Example 5.1. For each $t \in [0,T]$, let σ_t be the Dirac-delta measure at t. Then for every $E \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned}\sigma_t(E) &= \begin{cases} 1\,, & t \in E \cap [0,T] \\ 0\,, & t \notin E \cap [0,T] \end{cases} \\ &= \chi_{E\cap[0,T]}(t).\end{aligned}$$

Hence $\sigma_t(E)$ is a $\mathcal{B}([0,T])$-measurable function for all $E \in \mathcal{B}(\mathbb{R})$, and $\int_0^T \|\sigma_t\| db(s) = b(T)$. Let for each $t \in [0,T]$, $\theta(t,\cdot) : \mathbb{R} \to \mathbb{C}$ be defined by

$$\theta(t,u) = \int_{\mathbb{R}} \exp\{iuv\} d\sigma_t(v) = \int_{\{t\}} \exp\{iuv\} d\sigma_t(v) = \exp\{itu\}.$$

By Theorem 4.6, the functional G_t on $C_{a,b}[0,T]$ defined by $G_t(x) = \theta(t,x(t)) = \exp\{itx(t)\}$ is an element of $\mathcal{F}(C_{a,b}[0,T])$ and by Theorem 4.9, the functional F on $C_{a,b}[0,T]$ defined by

$$F(x) = \exp\left\{\int_0^T \exp\{itx(t)\}dt\right\} = 1 + \sum_{m=1}^{\infty}\left[\int_0^T \exp\{itx(t)\}dt\right]^m$$

is an element of $\mathcal{F}(C_{a,b}[0,T])$.

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