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HYERS-ULAM STABILITY OF CUBIC-QUARTIC FUNCTIONAL EQUATIONS ON RANDOM NORMED SPACES

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ABSTRACT. We introduce mixed cubic-quartic functional equations. And using the fixed point method, we prove the generalized Hyers-Ulam stability of cubic-quartic functional equations on random normed spaces.

1. INTRODUCTION

In almost all areas of mathematical analysis, we can raise the following fundamental question: When is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly? If we turn our attention to the case of functional equations, we can particularly ask the question when the solutions of an equation differing slightly from a given one must be close to the solution of the given equation.

The stable problem of functional equations is originated from such a fundamental question. In connection with the above question, S. M. Ulam [28] raised a question concerning the stability of homomorphisms:

Let G_1 be a group and G_2 a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$? Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [21]

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has provided a lot of influence in the development of what we call the *generalized Hyers-Ulam stability* or the *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The additive functional equation

$$f(x+y) = f(x) + f(y)$$

is one of the most famous functional equations, which is called the Cauchy equation. Every solution of the additive equation is called an additive function. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic function.* A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [24] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. Czerwik [4] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [10], [14]–[16], [19]–[23]). In the present paper, we investigate the generalized Hyers-Ulam stability for the mixed cubic-quartic functional equations in random normed spaces.

Jun and Kim [13] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

and they established the general solution and the generalized Hyers-Ulam stability for the cubic functional equation. The function $f(x) = x^3$ satisfies the above functional equation, which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic mapping. The oldest quartic functional equation was introduced by J.M.Rassias [?]:

$$f(x+2y) + f(x-2y) = 4(f(x+y) + f(x-y)) + 24f(y) - 6f(x)$$

In fact they proved that a mapping f between real vector spaces X and Y is a solution of the above equation if and only if there exists a unique symmetric multi-additive mapping $Q : X^4 \to Y$ such that $f(x) = Q(x,x,x,x)$ for all x. It is easy to show that the function $f(x) = x^4$ satisfies the above functional equation, which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.

We deal with the following functional equation deriving from quartic and cubic functions:

$$\begin{aligned} 4(f(3x+y)+f(3x-y)) &= 12(f(2x+y)+f(2x-y)) \\ &\quad -12(f(x+y)+f(x-y)) \\ &\quad -8f(y)-192f(x)+f(2y)+30f(2x) \end{aligned}$$

It is easy to see that the function $f(x) = ax^4 + bx^3$ is a solution of the functional equation. Gordji, Ebadian and Zolfaghari [5] investigated the general solution and the generalized Hyers-Ulam stability of the cubic-quartic functional equation.

The generalized Hyers-Ulam stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in([18], [15]). It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm T_M. In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [25, 26, 27].

We regard that Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [12]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n} x_i$ is defined recurrently by $T_{i=1}^{1} x_i = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i-1}$. It is known ([26]) that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$

Definition 1.2. A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

(RN_1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN_2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;

(RN_3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Let $(X, \|.\|)$ be a normed space. Then we can define a random normed space (X, μ, T_M) as follows:

$$\mu_x(t) = \frac{t + a\|x\|}{t + b\|x\|}$$

for all $t > 0$, for $b > a \geq 0$, and T_M is the minimum t-norm.

Definition 1.3. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) A RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4 ([26]). *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Theorem 1.5 ([27]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) the sequence $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

2. The Stability of the Mixed Cubic-Quartic Functional Equation: Odd Case

Through this paper let (Y, μ, T) be a complete RN-space. For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) &= 4(f(3x+y)+f(3x-y)) - 12(f(2x+y) \\ &\quad +f(2x-y)) + 12(f(x+y)+f(x-y)) - f(2y) \\ &\quad +8f(y) - 30f(2x) + 192f(x) \end{aligned}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of functional equation $Df(x,y) = 0$: odd case.

Theorem 2.1. *Let* X *be a linear space and* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \leq \frac{L}{8} \varphi(2x, 2y)$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \varphi(x,y)} \tag{2.1}$$

for all $x, y \in X$ *and all* $t > 0$. *Then* $C(x) := \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ *exists for each* $x \in X$ *and defines a cubic mapping* $C : X \to Y$ *such that*

$$\mu_{f(x)-C(x)}(t) \geq \frac{(8-8L)t}{(8-8L)t + L\varphi(0,x)} \tag{2.2}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Letting $x = 0$ in (2.1), we get

$$\mu_{f(2y)-8f(y)}(t) \geq \frac{t}{t + \varphi(0,y)}$$

for all $y \in X$ and all $t > 0$.

Replacing y by x in the above formula, we get

$$\mu_{f(2x)-8f(x)}(t) \geq \frac{t}{t + \varphi(0,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$. Let $f : X \to Y$ be a mapping and consider the set

$$S := \{f : X \to Y\}$$

and introduce the generalized metric on S:

$$d(f,h)=\inf\left\{\nu\in\mathbb{R}_+ : \mu_{f(x)-h(x)}(\nu t)\ge\frac{t}{t+\varphi(0,x)},\ \forall x\in X,\forall t>0\right\},$$

where, as usual, $\inf\phi=+\infty$. Then (S,d) is a complete generalized metric space(See [2]).

And we consider the linear mapping $J:S\to S$ such that

$$Jf(2x):=8f(x)$$

for all $x\in X$. Let $f,h\in S$ be given such that $d(f,h)=\varepsilon$. Then

$$\mu_{f(x)-h(x)}(\varepsilon t)\ge\frac{t}{t+\varphi(0,x)}$$

for all $x\in X$ and all $t>0$. Hence

$$\begin{aligned}\mu_{Jf(2x)-Jh(2x)}(L\varepsilon t) &= \mu_{8f(x)-8h(x)}(L\varepsilon t)\\ &= \mu_{f(x)-h(x)}\left(\frac{L}{8}\varepsilon t\right)\\ &\ge \frac{\frac{Lt}{8}}{\frac{Lt}{8}+\varphi(0,x)}\\ &\ge \frac{\frac{Lt}{8}}{\frac{Lt}{8}+\frac{L}{8}\varphi(0,2x)}\\ &= \frac{t}{t+\varphi(0,2x)}\end{aligned}$$

for all $x\in X$ and all $t>0$. So $d(f,h)=\varepsilon$ implies that $d(Jf,Jh)\le L\varepsilon$. This means that

$$d(Jf,Jh)\le Ld(f,h)$$

for all $f,h\in S$.

It follows from (2.3) that

$$\mu_{f(x)-8f(\frac{x}{2})}\left(\frac{L}{8}t\right)\ge\frac{t}{t+\varphi(0,x)}$$

for all $x\in X$ and all $t>0$. So $d(f,Jf)\le\frac{L}{8}$.

By Theorem 1.5, there exists a mapping $C:X\to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C(2x)=8C(x) \tag{2.4}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{f \in S : d(f, h) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.4) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{f(x)-C(x)}(\nu t) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(f, C) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{L}{8 - 8L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$\mu_{8^n Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}(8^n t) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{8^n Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}(t) \geq \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\mu_{DC(x,y)}(t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \to Y$ is cubic, as desired. □

Corollary 2.2. *Let X be a normed vector space with norm $\|\cdot\|$ and let $\theta \geq 0$ and p be a real number with $p > 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.5}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \frac{(2^p - 8)t}{(2^p - 8)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 2.3. *Let X be a linear space and let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.1). Then

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$$

exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + \varphi(0, x)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Consider the linear mapping $J : S \to S$ such that

$$Jf(x) := \frac{1}{8} f(2x)$$

for all $x \in X$.

Let $f, h \in S$ be given such that $d(f, h) = \varepsilon$. Then

$$\mu_{f(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jf(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{8}f(2x)-\frac{1}{8}h(2x)}(L\varepsilon t) \\
&= \mu_{f(2x)-h(2x)}(8L\varepsilon t) \\
&\geq \frac{8Lt}{8Lt + \varphi(0, 2x)}
\end{aligned}$$

$$\begin{aligned} &\geq \frac{8Lt}{8Lt + 8L\varphi(0,x)} \\ &= \frac{t}{t + \varphi(0,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(f,h) = \varepsilon$ implies that $d(Jf, Jh) \leq L\varepsilon$. This means that

$$d(Jf, Jh) \leq Ld(f,h)$$

for all $f, h \in S$.

It follows from (2.3) that

$$\mu_{f(x)-\frac{1}{8}f(2x)}\left(\frac{1}{8}t\right) \geq \frac{t}{t + \varphi(0,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{8}$.

By Theorem 1.5, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C(2x) = 8C(x)$$

for all $x \in X$. Since $f : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{f \in S : d(f,h) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.4) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{f(x)-C(x)}(\nu t) \geq \frac{t}{t + \varphi(0,x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{8^n} f(2^n x) = C(x)$$

for all $x \in X$;

(3) $d(f, C) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{1}{8 - 8L}.$$

This implies that the inequality (2.6) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let X be a normed vector space with norm $\|\cdot\|$, and let $\theta \geq 0$ and p be a real number with $0 < p < 3$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then $C(x) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(x)-C(x)}(t) \geq \frac{(8-2^p)t}{(8-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

3. The Stability of the Mixed Cubic-Quartic Functional Equation: Even Case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete random normed spaces: an even case.

Theorem 3.1. *Let X be a linear space and let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{16}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then $Q(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{(16-16L)t}{(16-16L)t + L\varphi(0,x)} \tag{3.1}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = 0$ in (2.1), we get

$$\mu_{f(2y)-16f(y)}(t) \geq \frac{t}{t+\varphi(0,y)}$$

for all $y \in X$ and all $t > 0$.

Replacing y by x in the above formula, we get

$$\mu_{f(2x)-16f(x)}(t) \geq \frac{t}{t+\varphi(0,x)} \tag{3.2}$$

for all $x \in X$ and all $t > 0$.

Let (S, d) be the generalized metric space defined by in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jf(2x) := 16f(x)$$

for all $x \in X$.

Let $f, h \in S$ be given such that $d(f, h) = \varepsilon$. Then

$$\mu_{f(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} \mu_{Jf(2x)-Jh(2x)}(L\varepsilon t) &= \mu_{16f(x)-16h(x)}(L\varepsilon t) \\ &= \mu_{f(x)-h(x)}\left(\frac{L}{16}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \varphi(0, x)} \\ &\geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \frac{L}{16}\varphi(0, 2x)} \\ &= \frac{t}{t + \varphi(0, 2x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(f, h) = \varepsilon$ implies that $d(Jf, Jh) \leq L\varepsilon$. This means that

$$d(Jf, Jh) \leq Ld(f, h)$$

for all $f, h \in S$.

It follows from (3.2) that

$$\mu_{f(x)-16f(\frac{x}{2})}\left(\frac{L}{16}t\right) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{16}$.

By Theorem 1.5, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(2x) = 16Q(x) \tag{3.3}$$

for all $x \in X$. Since $g : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{f \in S : d(f, h) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.3) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{f(x)-Q(x)}(\nu t) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{16 - 16L}.$$

This implies that the inequality (3.1) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let X be a normed vector space with norm $\|\cdot\|$ and let $\theta \geq 0$ and p be a real number with $p > 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then $Q(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \frac{(2^p - 16)t}{(2^p - 16)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{4-p}$ and we get the desired result. □

Theorem 3.3. *Let X be a linear space and let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then

$$Q(x) := \lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{(16 - 16L)t}{(16 - 16L)t + \varphi(0, x)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jf(x) := \frac{1}{16} f(2x)$$

for all $x \in X$.

Let $f, h \in S$ be given such that $d(f, h) = \varepsilon$. Then

$$\mu_{f(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} \mu_{Jf(x)-Jh(x)}(L\varepsilon t) &= \mu_{\frac{1}{16}f(2x)-\frac{1}{16}h(2x)}(L\varepsilon t) \\ &= \mu_{f(2x)-h(2x)}(16L\varepsilon t) \\ &\geq \frac{16Lt}{16Lt + \varphi(0, 2x)} \\ &\geq \frac{16Lt}{16Lt + 16L\varphi(0, x)} \\ &= \frac{t}{t + \varphi(0, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(f, h) = \varepsilon$ implies that $d(Jf, Jh) \leq L\varepsilon$. This means that

$$d(Jf, Jh) \leq L d(f, h)$$

for all $g, h \in S$.

It follows from (3.2) that

$$\mu_{f(x)-\frac{1}{16}f(2x)}\left(\frac{1}{16}t\right) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{1}{16}$.

By Theorem 1.5, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(2x) = 16Q(x)$$

for all $x \in X$. Since $g : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{f \in S : d(f, h) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.3) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{f(x)-Q(x)}(\nu t) \geq \frac{t}{t + \varphi(0, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} \frac{1}{16^n} f(2^n x) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{16 - 16L}.$$

This implies that the inequality (6.10) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let X be a normed vector space with norm $\|\cdot\|$ and let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then $Q(x) := \lim_{n \to \infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \frac{(16 - 2^p)t}{(16 - 2^p)t + 2\theta \|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-4}$ and we get the desired result. □

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