

## DIGITAL OPTION PRICING BASED ON COPULAS WITH STOCHASTIC SIMULATION

M. S. KIM<sup>a</sup> AND SEKI KIM<sup>b,\*</sup>

**ABSTRACT.** In this paper, we show the effectiveness of copulas by comparing the correlation of market data of year 2010 with those of years 2006-2009 and investigate copula functions as pricing methods of digital and rainbow options through real market data. We propose an accurate method of pricing rainbow options by using the correlation coefficients obtained from the copula functions depending on strike prices between assets instead of simple traditional correlation coefficients.

### 1. INTRODUCTION

Copula functions represent a methodology which has recently become the most significant new tool to handle in a flexible way the co-movement between markets, risk factors and other relevant variables studied in finance. While the tool is borrowed from the theory of statistics, it has been gathering more and more popularity both among academics and practitioners in the field of finance principally because of the intimate increase of volatility and erratic behavior of financial markets [2]. These new developments have caused standard tools of financial mathematics. The need to reach effective diversification has led to new investment products, bound to exploit the credit risk features of the assets. It is particularly for the evaluation of these new products, such as securitized assets and basket credit derivatives that the need to account for co-movement among non-normally distributed variables has become an unavoidable task.

Copula functions have been applied to the solution of these problems in mathematical finance. In fact, the use of copula functions enables the task of specifying the marginal distribution to be decoupled from the dependence structure of variables.

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\*Corresponding author.

For instance, the fact that the forward-futures spread is more significant for long maturity is shown by evaluating the spread based on copula models [8]. For applying copula functions, we have to settle three problems which are 1) what is more proper in Gaussian or Archimedean copula functions and 2) how to determine marginal distributions about real data, finally 3) how to select kinds of Archimedean copula functions [4]. In this paper, they are used in numerical examples that illustrate the use of the copula functions in the digital and rainbow option pricing.

This paper is organized as follows. Section 2 provides an overview of copula functions. In Section 3, copula algorithm method as well as digital option pricing and rainbow option pricing is established and shown for numerical experiments. In Section 4, stochastic simulations are performed and investigated through copula models. The conclusion of this work is in Section 5.

## 2. PRELIMINARIES

The analysis of copula functions started from the approach suggested by Nelsen. Cherubini, Luciano, and Vecchiato applied copula functions to finance fields. In this section, we introduce the fundamental concept of copula and some knowledge needed.

### 2.1. Copula

**Definition 2.1** ([6]). A two-dimensional subcopula is a function  $C'$  with the following properties:

1.  $Dom C' = S_1 \times S_2$  where  $S_1$  and  $S_2$  are subsets of  $I$  containing 0 and 1.
2.  $C'$  is grounded and 2-increasing.
3. For every  $u$  in  $S_1$  and every  $v$  in  $S_2$ ,

$$C'(u, 1) = u, \quad C'(1, v) = v$$

Note that for every  $(u, v)$  in  $Dom C'$ ,  $0 \leq C'(u, v) \leq 1$  so that  $Ran C'$  is also a subset of  $I$  (i.e.  $[0, 1]$ ).

**Definition 2.2** ([6]). A two-dimensional copula is a function  $C$  whose domain  $I^2$  with the following properties:

1. For every  $u, v$  in  $I$

$$C(u, 1) = u, \quad C(1, v) = v \quad \text{and} \quad C(u, 0) = 0 = C(0, v)$$

2. For every  $u_1, u_2, v_1, v_2$  in  $I$  such that  $u_1 \leq u_2$  and  $v_1 \leq v_2$

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$$

## 2.2. Sklar's Theorem

**Lemma 2.3** ([6]). *Let  $H$  be a joint distribution with margins  $F$  and  $G$ , then there is a unique subcopula  $C'$  such that*

1.  $\text{Dom } C' = \text{Ran } F \times \text{Ran } G$
2.  $H(x, y) = C'(F(x), G(y))$  for all  $x, y$ .

**Lemma 2.4** ([6]). *Let  $C'$  be a subcopula. Then exists a copula  $C$  s.t*

$$C(u, v) = C'(u, v) \quad \text{for all } (u, v) \text{ in } \text{Dom } C'.$$

**Theorem 2.5** ([6]). (Sklar's Theorem) *Let  $H$  be a joint distribution function with margins  $F$  and  $G$  then there exists a copula  $C$  such that*

$$H(x, y) = C(F(x), G(y)) \quad \text{for all } x, y.$$

If  $F$  and  $G$  are continuous, then  $C$  is unique. That is,  $C$  is uniquely determined on  $\text{Ran } F \times \text{Ran } G$ . Conversely, if  $C$  is a copula and  $F$  and  $G$  are distribution functions, then  $H$  is a joint distribution function.

**Corollary 2.6** ([6]). *Let*

*$H, F, G$ , and  $C'$  be joint distribution functions,  
 $F^{(-1)}$  and  $G^{(-1)}$  be quasi-inverses of  $F$  and  $G$  respectively.*

*Then*

$$C'(u, v) = H(F^{(-1)}(u), G^{(-1)}(v)) \quad \text{for any } (u, v) \text{ in } \text{Dom } C'$$

**2.3. Survival Copulas** In many applications, the random variables of interest represent the lifetimes of individuals or objects in some population. The probability of an individual living or surviving beyond time  $x$  is given by the *survival function*[6]

$$\bar{F}(x) = P[X > x] = 1 - F(x)$$

The *joint survival function*  $\bar{H}$  for the joint distribution function  $H$  is given by

$$\bar{H}(x, y) = P[X > x, Y > y]$$

Suppose that the copula of  $X$  and  $Y$  is  $C$ . Then

$$\bar{H}(x, y) = \hat{C}(\bar{F}(x), \bar{G}(y))$$

with defining a function  $\widehat{C}$  (*survival copula*) from  $I^2$  to  $I$  by

$$\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

Note that  $\overline{C}(u, v) = P[U > u, V > v] = 1 - u - v + C(u, v) = \widehat{C}(1 - u, 1 - v)$ .

**2.4. Upper and Lower tail dependence** Tail dependence can be measured and relates the amount of dependence in the upper right quadrant tail or lower left one of a bivariate distribution[6].

Upper tail dependence is defined as:

$$\lambda_{upper} = \lim_{u \rightarrow 1} P[Y \geq F_Y^{-1}(u) | X \geq F_X^{-1}(u)]$$

where  $F^{-1}$  denotes the inverse cumulative distribution function and  $u$  is an uniform variable defined over  $(0, 1)$ .

Since

$$P(X > a, Y > b) = 1 - P(X \leq a) - P(Y \leq b) + P(X \leq a, Y \leq b),$$

$P[Y \geq F_Y^{-1}(u) | X \geq F_X^{-1}(u)]$  can be written as below:

$$\frac{1 - P[X \leq F_X^{-1}(u)] - P[Y \leq F_Y^{-1}(u)] + P[X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u)]}{1 - P[X \leq F_X^{-1}(u)]}.$$

From the fact that  $F_X(x) = P[X \leq x]$  and  $F_X(F_X^{-1}(t)) = t$ , we have

$$P[X \leq F_X^{-1}(u)] = P[Y \leq F_Y^{-1}(u)] = u,$$

and

$$P[X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u)] = C(u, u).$$

Hence the upper tail dependence can be rewritten as:

$$\lambda_{upper} = \lim_{u \rightarrow 1} \frac{1 - 2u + C(u, u)}{1 - u}$$

Lower tail dependence is symmetrically defined:

$$\lambda_{lower} = \lim_{u \rightarrow 0} P[Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)]$$

Therefore we obtain the lower tail dependence as below:

$$\lambda_{lower} = \lim_{u \rightarrow 0} \frac{C(u, u)}{u}$$

## 2.5. Copula Functions

### 2.5.1. Archimedean copula

**Definition 2.7** ([6]). Let  $\varphi$  be a continuous, strictly decreasing function from  $I$  to  $[0, \infty)$  such that  $\varphi(1) = 0$ . We define that  $\varphi^{[-1]}$ : the pseudo-inverse of  $\varphi$  with  $Dom \varphi^{[-1]} = [0, \infty)$  and  $Ran \varphi^{[-1]} = I$  where

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t < \infty \end{cases}.$$

Note that  $\varphi^{[-1]}$  is continuous and nonincreasing on  $[0, \infty]$ , and strictly decreasing on  $[0, \varphi(0)]$ . Furthermore,  $\varphi^{[-1]}(\varphi(u)) = u$  on  $I$ , and

$$\begin{aligned} \varphi(\varphi^{[-1]}(t)) &= \begin{cases} t, & 0 \leq t \leq \varphi(0) \\ \varphi(0), & \varphi(0) \leq t \leq \infty \end{cases} \\ &= \min(t, \varphi(0)). \end{aligned}$$

Finally, if  $\varphi(0) = \infty$ , then  $\varphi^{[-1]} = \varphi^{-1}$ .

**Lemma 2.8** ([6]). Let  $\varphi$  be a continuous, strictly decreasing function from  $I$  to  $[0, \infty]$  such that  $\varphi(1) = 0$ , and let  $\varphi^{[-1]}$  be the pseudo-inverse of  $\varphi$ .

Let  $C$  be the function from  $I^2$  to  $I$  given by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)).$$

Then  $C$  satisfies the boundary conditions for a copula.

i.e.,  $C(u, 1) = u$ ,  $C(1, v) = v$  and  $C(u, 0) = 0 = C(0, v)$ .

**2.5.2. Important Examples of Archimedean copulas** Let  $\varphi$  be a continuous, strictly decreasing function from  $I$  to  $[0, \infty]$  such that  $\varphi(1) = 0$ , and let  $\varphi^{[-1]}$  be the pseudo-inverse of  $\varphi$  [6].

Let  $C$  be the function from  $I^2$  to  $I$  given by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)).$$

Examples of bivariate Archimedean copulas are as the followings [5]:

- Clayton copula(1978)

$$C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad \varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1), \quad \theta \geq -1 \text{ (except 0)}.$$

- Gumbel copula(1960)

$$C(u, v) = e^{-[(-\ln u)^\theta + (-\ln v)^\theta]^{\frac{1}{\theta}}}, \quad \varphi(t) = (-\ln t)^\theta, \quad \theta \geq 1.$$

· Frank copula(1979)

$$C(u, v) = -\frac{1}{\theta} \ln \left( 1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right), \quad \varphi(t) = -\ln \left( \frac{e^{-\theta t} - 1}{e^{-\theta} - 1} \right), \quad \theta (\neq 0) \in R.$$

**2.6. Digital Option** In finance, a digital[binary] option is a type of the option in which the payoff is either some fixed amount of some asset or nothing at all [3]. To give an intuitive grasp of the use of copula functions in finance, consider a very simple product, a bivariate digital option. This option pays one unit of currency if two stocks or indexes are above or below a pair of strike price levels. Options like these are very often used in structured finance, particularly index-linked products [2].

For example, assume the option of companies  $A$  and  $B$  has same exercise date  $T$  and strike prices  $K_A$  and  $K_B$  respectively, then the price of this digital put option[ $DP$ ] in a complete market is

$$DP = \exp[-r(T - t)]Q(K_A, K_B)$$

where  $Q(K_A, K_B)$  is the joint risk-neutral probability that both A and B market indexes are below the corresponding strike prices. In order to compare the price of our bivariate product with that of the univariate ones, it would be great if we could write the price as

$$DP = \exp[-r(T - t)]Q(K_A, K_B) = \exp[-r(T - t)]C[Q(K_A), Q(K_B)].$$

Consider a bivariate digital call option[ $DC$ ]. Differently from the digital put option, it pays one unit of currency if both A and B indexes are above the strike levels  $K_A$  and  $K_B$ . The relevant probability in this case is

$$DC = \exp[-r(T - t)]\overline{Q}(K_A, K_B).$$

Like  $DP$ , we can represent this as

$$DC = \exp[-r(T - t)]\overline{Q}(K_A, K_B) = \exp[-r(T - t)]\widehat{C}(\overline{Q}(K_A), \overline{Q}(K_B))$$

where  $\widehat{C}$  is a survival copula. Using survival copulas, it can be rewritten as belows

$$\widehat{C}(\overline{Q}(K_A), \overline{Q}(K_B)) = 1 - Q(K_A) - Q(K_B) + C(Q(K_A), Q(K_B)).$$

For applying copula functions, we can use Archimedean copulas to obtain

$$C(Q(K_A), Q(K_B)) = \varphi^{[-1]}(\varphi(Q(K_A)) + \varphi(Q(K_B)))$$

where  $\varphi$  is the generator function of Archimedean copula functions. Finally we may price our bivariate claim using

- Digital Put option(DP)

$$DP = \exp[-r(T - t)]\varphi^{[-1]}(\varphi(Q(K_A)) + \varphi(Q(K_B))).$$

- Digital Call option(DC)

$$DC = \exp[-r(T - t)](1 - Q(K_A) - Q(K_B) + \varphi^{[-1]}(\varphi(Q(K_A)) + \varphi(Q(K_B)))).$$

### 3. METHOD ALGORITHM

#### 3.1. Copula Algorithm

**Algorithm 3.1** ([4]). :

1. Simulate two independent  $U(0, 1)$  random variates  $s$  and  $q$ .
2. Set  $t = K_{copula}^{-1}(q)$ , where  $K_{copula}$  is the distribution function  $C(u, v)$ .
3. Set  $u = \varphi^{-1}(s\varphi(t))$  and  $v = \varphi^{-1}((1 - s)\varphi(t))$

For each Archimedean copula we need the follwongs with Kendall’s  $\tau$ :

B	C	D	E	F	G
$\theta$	Generator $\varphi(t)$	$\varphi'(t)$	Inverse $\varphi^{-1}(t)$	$K_{copula} = t - \frac{\varphi(t)}{\varphi'(t)}$	inverse $K_{copula}^{-1}$

In Gumbel copula, we have a generator function  $\varphi(t) = (-\ln t)^\theta$ :

B	C	D	E	F	G
$\theta = \frac{1}{1-\tau}$	$(-\ln t)^\theta$	$-\theta(\ln t)^{\theta-1}\frac{1}{t}$	$e^{-t^{\frac{1}{\theta}}}$	$t - \frac{(t \ln t)}{\theta}$	$-\frac{\ln t}{\theta} - \frac{1}{\theta} + 1$

Therefore, we can obtain  $u$  and  $v$  as below:

$$u = e^{-(s(-\ln t)^\theta)^{\frac{1}{\theta}}}, \quad v = e^{-((1-s)(-\ln t)^\theta)^{\frac{1}{\theta}}}.$$

In Clayton copula, with a generator function  $\varphi(t) = t^{-\theta} - 1$ :

B	C	D	E	F	G
$\theta = \frac{2\tau}{1-\tau}$	$t^{-\theta} - 1$	$-\theta t^{-\theta-1}$	$(1 + \theta)^{-\frac{1}{\theta}}$	$t - \frac{t^{\theta+1}-t}{\theta}$	$-\frac{t^\theta(\theta+1)}{\theta} + \frac{1}{\theta} + 1$

Similarly, we have

$$u = (1 + (s(t^{-\theta} - 1)))^{-\frac{1}{\theta}}, \quad v = (1 + ((1 - s)(t^{-\theta} - 1)))^{-\frac{1}{\theta}}.$$

**3.2. Digital Option Pricing** Consider two univariate digital put options with the maturity  $T$  and strike prices  $K_1$  and  $K_2$  respectively. If we denote by  $X_T$  and  $Y_T$  the prices of the underlying assets at maturity then we have

$$P(X_T \leq K_1) = F_1(K_1), \quad P(Y_T \leq K_2) = F_2(K_2).$$

The price of the bivariate digital put option can be evaluated by the following formula.

$$P(X_T \leq K_1, Y_T \leq K_2) = H(K_1, K_2).$$

According to Sklar's theorem, we have

$$H(K_1, K_2) = C(F_1(K_1), F_2(K_2)).$$

Assume that asset value random processes  $X$  and  $Y$  have lognormal distributions under a constant interest rate  $r$  and constant volatilities  $\sigma_X$  and  $\sigma_Y$  respectively as in the Black-Scholes model[1], then we have the individual prices of Put Options,

$$P(X_T \leq K_1) = F_1(K_1) = N(-d_{2X}(K_1)) \equiv N\left(-\frac{\ln(\frac{X_0}{K_1}) + (r - \frac{1}{2}\sigma_X^2)T}{\sigma_X\sqrt{T}}\right),$$

and Call Options,

$$P(Y_T > K_2) = \bar{F}_2(K_2) = N(d_{2Y}(K_2)) \equiv N\left(\frac{\ln(\frac{Y_0}{K_2}) + (r - \frac{1}{2}\sigma_Y^2)T}{\sigma_Y\sqrt{T}}\right)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{s^2}{2}} ds \quad \text{and} \quad d_{2Z}(K) = \frac{\ln(\frac{Z_0}{K}) + (r - \frac{1}{2}\sigma_Z^2)T}{\sigma_Z\sqrt{T}}.$$

with  $X_0$  and  $Y_0$ , are initial values of underlying assets. Hence the value of digital put option can be written as

$$P(X_T \leq K_1, Y_T \leq K_2) = H(K_1, K_2) = C(F_1(K_1), F_2(K_2)) = C(N(-d_{2X}(K_1)), N(-d_{2Y}(K_2))),$$

and the value of digital call option should be

$$P(X_T > K_1, Y_T > K_2) = \bar{H}(K_1, K_2) = \hat{C}(\bar{F}_1(K_1), \bar{F}_2(K_2)) = \hat{C}(N(d_{2X}(K_1)), N(d_{2Y}(K_2))).$$

Consequently, we show the price of digital put option using Clayton copula among Archimedean copula functions as the following.

$$C(N(-d_{2X}(K_1)), N(-d_{2Y}(K_2))) = (N(-d_{2X}(K_1))^{-\theta} + N(-d_{2Y}(K_2))^{-\theta} - 1)^{-\frac{1}{\theta}}.$$

Similarly, we have for the call option value as below,

$$\begin{aligned}\widehat{C}(N(d_{2X}(K_1)), N(d_{2Y}(K_2))) &= \overline{C}(N(-d_{2X}(K_1)), N(-d_{2Y}(K_2))) \\ &= 1 - N(-d_{2X}(K_1)) - N(-d_{2Y}(K_2)) \\ &\quad + (N(-d_{2X}(K_1))^{-\theta} + N(-d_{2Y}(K_2))^{-\theta} - 1)^{-\frac{1}{\theta}}\end{aligned}$$

where  $\theta$  is a Clayton copula parameter and can be estimated using Kendall's  $\tau[2]$ .

**3.3. Rainbow Option Pricing** A rainbow option is a derivative whose value is depending on two or more underlying securities or events. Rainbow options are usually calls or puts on the best or worst of underlying assets, or options which pay the best or worst of assets. The number of assets underlying the option is called the number of colors of the rainbow. We consider a rainbow call option corrected formula based on the minimum value of two underlying assets  $X$  and  $Y$  at maturity  $T$  over strike price  $K$ . Rainbow call option price under the risk-neutral measure is shown as below[9].

$$\begin{aligned}C(X, Y, K) &= e^{-rT} \mathbb{E}[\max(\min(X_T, Y_T) - K, 0)] \\ &= Y_0 N_2 \left( d_{2Y}(K) + \sigma_Y \sqrt{T}, \frac{\ln(X_0/Y_0) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}, \frac{\rho \sigma_X - \sigma_Y}{\sigma} \right) \\ &\quad + X_0 N_2 \left( d_{2X}(K) + \sigma_X \sqrt{T}, \frac{\ln(Y_0/X_0) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}, \frac{\rho \sigma_Y - \sigma_X}{\sigma} \right) \\ &\quad - K e^{-rT} N_2(d_{2Y}(K), d_{2X}(K), \rho)\end{aligned}$$

where

$$N_2(\alpha, \beta, \rho) = \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} \frac{1}{2\pi \sqrt{1-\rho^2}} e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} dx dy,$$

with  $\rho$  which is the correlation coefficient between two assets and  $\sigma^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$  with two asset volatilities  $\sigma_X$  and  $\sigma_Y$  respectively.

It can be shown that the rainbow call option value turns out to be equal to the value of an European call option if two assets are definitely same i.e  $X \equiv Y$ .

**Proposition 3.1.** *The rainbow call option formula becomes the formula for European call option values if two underlying assets are definitely same.*

*Proof.* First, with  $X = Y$ , we have  $\sigma = \sqrt{2(1-\rho)\sigma_X^2}$  since  $\sigma_X = \sigma_Y$ . This shows

$$\theta = \frac{\rho\sigma_X - \sigma_Y}{\sigma} = \frac{(\rho - 1)\sigma_X}{\sqrt{2(1 - \rho)\sigma_X^2}} = -\frac{\sqrt{1 - \rho}}{\sqrt{2}} \rightarrow 0$$

as  $\rho \rightarrow 1$ . Now from the first part of three terms of the rainbow formula,

$$\begin{aligned} & \lim_{\rho \rightarrow 1} Y_0 N_2 \left( d_{2Y}(K) + \sigma_Y \sqrt{T}, \frac{\ln(X_0/Y_0) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}, \frac{\rho\sigma_X - \sigma_Y}{\sigma} \right) \\ &= Y_0 \lim_{\rho \rightarrow 1} \int_{-\infty}^{\frac{\ln(Y_0/K) + (r + \frac{1}{2}\sigma_Y^2)T}{\sigma_Y \sqrt{T}}} \int_{-\infty}^{\frac{-\frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}} \frac{1}{2\pi \sqrt{1 - \theta^2}} e^{-\frac{x^2 + y^2 - 2\theta xy}{2(1 - \theta^2)}} dx dy \end{aligned}$$

where  $\frac{-\frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} = -\frac{1}{2}\sigma \sqrt{T} \rightarrow 0$  as  $\rho \rightarrow 1$  since  $\sigma$  goes to 0 as  $\rho \rightarrow 1$ .

$$\begin{aligned} &= Y_0 \int_{-\infty}^{\frac{\ln(Y_0/K) + (r + \frac{1}{2}\sigma_Y^2)T}{\sigma_Y \sqrt{T}}} \int_{-\infty}^0 \frac{1}{2\pi} e^{-\frac{x^2 + y^2}{2}} dx dy \\ &= Y_0 \int_{-\infty}^{\frac{\ln(Y_0/K) + (r + \frac{1}{2}\sigma_Y^2)T}{\sigma_Y \sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ &= \frac{1}{2} Y_0 N(d_{2Y}(K) + \sigma_Y \sqrt{T}). \end{aligned}$$

From the second part, similarly

$$\begin{aligned} & \lim_{\rho \rightarrow 1} X_0 N_2 \left( d_{2X}(K) + \sigma_X \sqrt{T}, \frac{\ln(Y_0/X_0) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}, \frac{\rho\sigma_Y - \sigma_X}{\sigma} \right) \\ &= X_0 \lim_{\rho \rightarrow 1} \int_{-\infty}^{\frac{\ln(X_0/K) + (r + \frac{1}{2}\sigma_X^2)T}{\sigma_X \sqrt{T}}} \int_{-\infty}^{\frac{-\frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}} \frac{1}{2\pi \sqrt{1 - \theta^2}} e^{-\frac{x^2 + y^2 - 2\theta xy}{2(1 - \theta^2)}} dx dy \\ &= X_0 \int_{-\infty}^{\frac{\ln(X_0/K) + (r + \frac{1}{2}\sigma_X^2)T}{\sigma_X \sqrt{T}}} \int_{-\infty}^0 \frac{1}{2\pi} e^{-\frac{x^2 + y^2}{2}} dx dy \\ &= \frac{1}{2} X_0 N(d_{2X}(K) + \sigma_X \sqrt{T}). \end{aligned}$$

Hence the sum of first two terms becomes

$$X_0 N \left( \frac{\ln(X_0/K) + (r + \frac{1}{2}\sigma_X^2)T}{\sigma_X \sqrt{T}} \right).$$

Now we investigate the last part of the formula.

$$\begin{aligned}
& \lim_{\rho \rightarrow 1} K e^{-rT} N_2(d_{2X}(K), d_{2Y}(K), \rho) \\
&= K e^{-rT} \lim_{\rho \rightarrow 1} \int_{-\infty}^{d_{2X}(K)} \int_{-\infty}^{d_{2Y}(K)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} dx dy \\
&= K e^{-rT} \int_{-\infty}^{\min(d_{2X}(K), d_{2Y}(K))} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} \lim_{\rho \rightarrow 1} \int_{-\infty}^{\frac{\max(d_{2X}(K), d_{2Y}(K))-\rho s}{\sqrt{1-\rho^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du ds \\
&= K e^{-rT} \int_{-\infty}^{\min(d_{2X}(K), d_{2Y}(K))} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds \\
&= K e^{-rT} N(d_{2X}(K)) \\
&= K e^{-rT} N\left(\frac{\ln(X_0/K) + (r - \frac{1}{2}\sigma_X^2)T}{\sigma_X\sqrt{T}}\right).
\end{aligned}$$

This completes the proof.  $\square$

The correlation between two assets is important to evaluate rainbow options especially depending on strike prices and it can be estimated from the copula functions established earlier on the relationship between assets. We propose an accurate method of pricing rainbow options by using the correlation coefficients obtained from the copula functions depending on strike prices between asstes instead of simple traditional correlation coefficients.

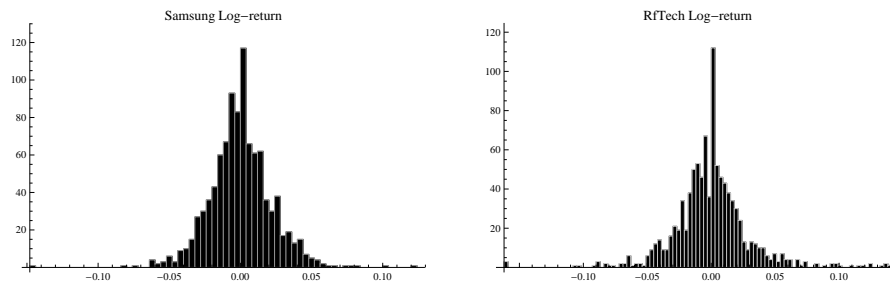
#### 4. NUMERICAL RESULTS

**4.1. Information of Stock Price** From 2005, cellular phone industry is not only for simple communication but also seeking for a wide range of development such as game, videotelephony, up-to-date smartphone. These days, one of top companies in cellular phone industry is Samsung Electronics. Therefore, it can be selected with relevant companies which are cooperative firms of Samsung Electronics. These are Abico, Displaytech, Sanyang, Rftech, Mtechvision. For the purpose of comparing non-subsidiary company or from different industry, LG Electronics, SK Hynix and CJ Corporation are investigated.

We consider the relationship between them and analyze how effect volatilities of stock prices make to pricing of financial derivatives with the relationship like correlation coefficients between them. We choose total 5 years(from 2006 to 2010) including around 1200 discrete data each company and compare data from the first 4 years with those of the last year. The volatility of stock prices during 2006-2009

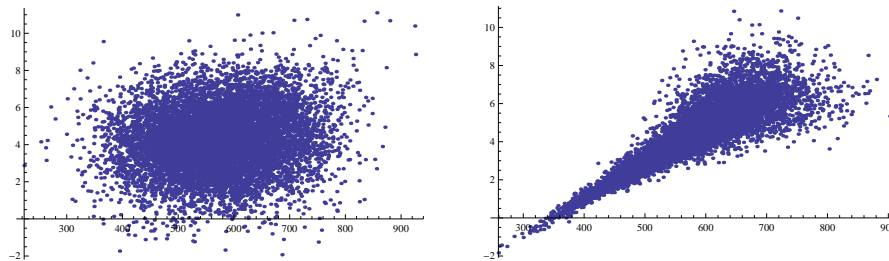
analyzed and estimated correlations between them are applied to investigate the financial stability condition of companies in 2010.

**4.2. Distribution and Correlation** We use the estimated distributions of selected companies from the real log-return data to find the parameters of relationship between them. The distributions of Samsung Electronics & Rftech obtained are approximately normal as shown in Figure 1.



*Figure 1.* Distributions of Samsung Electronics & Rftech

For the sake of choosing apt copula functions, the calculated values of upper and lower tail dependences which determine the sensitivity of heavy tail are given by the formulae in Section 2.4 as 0.115 and 0.991 respectively for Samsung-Rftech and 0.182 and 0.992 respectively for Samsung-Displaytech. Therefore, we can use rather Clayton copula for adequate relationship because of the lower tail dependence value which is not close to zero. The result after performing simulations with Gumbel and Clayton copulas for Samsung-Rftech is shown in Figure 2.



*Figure 2.* Gumbel(Left) & Clayton(Right) for Samsung-Rftech

The correlation coefficients between two companies found by copula functions can be applied to expect the behavior of financial stability of firms. Dynamic fluctuation in curve of correlation coefficients between firms represents the active economic

behavior while damping to less fluctuatedly static curve in correlation coefficients leads us to expect the financial problem of one of companies compared. This is another advantage in applications of copula functions.

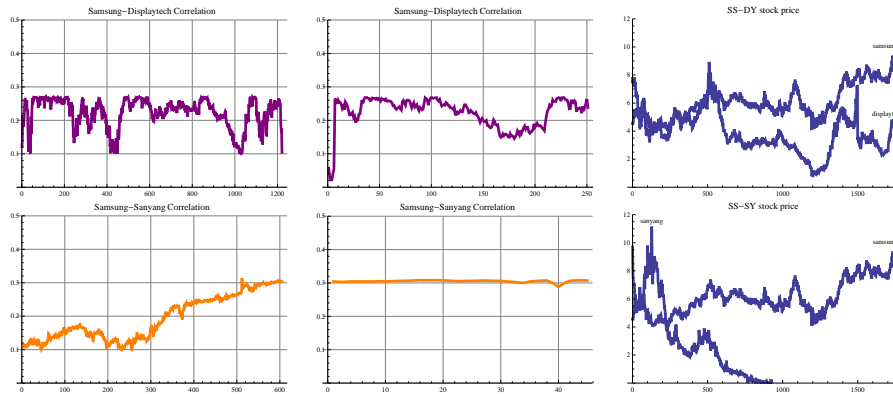


Figure 3. Samsung-Displaytech(1st row) & Samsung-Sanyang(2nd row)

In Figure 3, we see the obvious result explained as the above. First two correlation coefficient curves each row include the 3-4 year coefficient curve for the first figure and the last year curve of 2009 or 2010 for the second. The third figure each row shows the dynamics of scaled stock prices of companies and it is worthy of notice that the company Sanyang was defaulted in 2009.

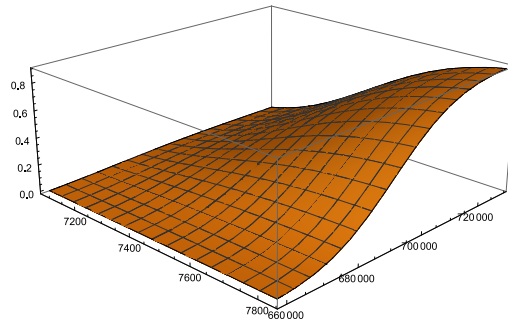
**4.3. Option Pricing** From real log-return data, we obtain estimated parameters for copula functions to establish copula models as Table 1.

Company	Kendall's $\tau$	Gumbel	Clayton
Sam-Rf	0.166387	1.1996	0.399194
Sam-Disp	0.20352	1.25552	0.511048
Sam-Abi	0.173348	1.2097	0.419398
Sam-LG	0.347775	1.53321	1.06643
Sam-SK	0.385126	1.62635	1.2527
LG-CJ	0.001171	1.00117	0.00234517

Table 1. Copula parameters between companies

As well as parameters for correlation between firms except almost independent relationship LG-CJ used to copula models we investigate, it is not difficult to find the volatility of each company for option pricing using real log return data. For

example, Samsung Electronics' volatility is  $\sigma_{ss} = 0.0215614$  with starting stock price  $S_0 = 659000$  and Rftech has  $\sigma_{rf} = 0.0328137$  as a volatility with  $S_0 = 7080$ . Now, the digital put option value for Samsung-Rftech is calculated using copula functions like Gumbel and Clayton, and Figure 4 shows option values depending on strike prices of Samsung and Rftech under Clayton copula function.



*Figure 4.* Digital Put Option Price of Samsung and Rftech

Conversely, the price of rainbow options can be estimated from the correlation coefficients depending on strike prices of these two firms which was calculated using proper copula functions. This is another accurate method to evaluate prices of rainbow options not using simple traditional correlation coefficients.

## 5. CONCLUSION

Upper and lower tail dependences which determine the sensitivity of heavy tail give the right choice for proper copula functions to show the adequate relationship between companies. We can select Clayton, Gumbel, or Frank copula for better representation of correlation between firms. We show the effectiveness of copulas by comparing the correlation of market data of year 2010 with those of years 2006-2009 and investigate copula functions as pricing methods of digital options through real market data. The correlation coefficients between two companies found by copula functions can be applied to expect the behavior of financial stability of firms. Dynamic fluctuation in curve of correlation coefficients between firms represents the active economic behavior while damping to less fluctuatedly static curve in correlation coefficients leads us to expect the financial problem of one of companies compared. This is an important advantage in applications of copula functions. Stochastic simulations for digital options are performed under the condition with adequately chosen

copula functions and estimated parameters from real market data. Moreover, the accurate method of pricing rainbow options is proposed using the correlation coefficients obtained from the copula functions depending on strike prices between assets instead of simple traditional correlation coefficients. This application can be extended to evaluate credit risk features and credit derivatives.

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<sup>a</sup>RWTH AACHEN UNIVERSITY, AACHEN, GERMANY  
Email address: kim@stochastik.rwth-aachen.de

<sup>b</sup>DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, KOREA  
Email address: skim@skku.edu