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DERIVATION AND ACTOR OF CROSSED POLYMODULES

Bijan Davvaz [a,*] and Murat Alp [b]

Abstract. An old result of Whitehead says that the set of derivations of a group with values in a crossed G-module has a natural monoid structure. In this paper we introduce derivation of crossed polymodule and actor crossed polymodules by using Lue's and Norrie's constructions. We prove that the set of derivations of a crossed polygroup has a semihypergroup structure with identity. Then, we consider the polygroup of invertible and reversible elements of it and we obtain actor crossed polymodule.

1. Introduction

The notion of crossed module was introduced by Whitehead in [19]. Later so many applications of crossed module has been presented such as actor crossed module, Crossed polymodule, etc. Actor crossed module was defined by Norrie in [16]. In Norrie' s paper the notion of derivation plays very important role to define actor crossed module, also see [15]. Derivations and Whitehead's groups of derivations, examples of derivations groups were presented by Gilbert in [12]. Crossed polymodules and some examples of crossed polymodules were introduced by Davvaz and Alp in [9]. In this paper, we use Norrie's way to define actor crossed polymodule. First we define a derivation and we prove some of its properties. Later, to define actor crossed polymodule we give action and boundary homomorphism together. Finally we present actor crossed polymodule.

Let G be a group and Ω be a non-empty set. A *(left) group action* is a function $\tau : G \times \Omega \to \Omega$ that satisfies the following two axioms:

(1) $\tau(gh, \omega) = \tau(g, \tau(h, \omega))$, for all $g, h \in G$ and $\omega \in \Omega$,

(2) $\tau(e, \omega) = \omega$, for all $\omega \in \Omega$.

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*Corresponding author.

For $\omega \in \Omega$ and $g \in G$, we write ${}^{g}\omega := \tau(g, \omega)$. A *crossed module* $X = (M, G, \partial, \tau)$ consists of groups M and G together with a homomorphism $\partial : M \to G$ and a (left) action $\tau : G \times M \to M$ on M, satisfying the conditions:

(1) $\partial({}^{g}m) = g\partial(m)g^{-1}$, for all $m \in M$ and $g \in G$,

(2) ${}^{\partial(m)}m' = mm'm^{-1}$, for all $m, m' \in M$.

In this paper, we introduce the derivation of crossed polymodule and actor crossed polymodules by using Norrie's way. In Section 2, we present some basic facts about polygroups that underlie the subsequent material. In Section 3, we present the definition of crossed polymodule and main theorem about the fundamental crossed module derived from a crossed polymodule. The main section of the paper is Section 4. In this section, we introduce the concept of derivation of crossed polymodules and we give some results in this respect. In particular, we prove that the set of derivations of a crossed polygroup has a semihypergroup structure with identity. Then, we consider the polygroup of invertible and reversible elements of it and we obtain actor crossed polymodule.

2. Basic Facts about Polygroups

The polygroup theory is a natural generalization of the group theory. In a group the composition of two elements is an element, while in a polygroup the composition of two elements is a set. Polygroups have been applied in many area, such as geometry, lattices, combinatorics and color scheme. There exists a rich bibliography: publications appeared within 2013 can be found in "Polygroup Theory and Related Systems" by B. Davvaz [4]. This book contains the principal definitions endowed with examples and the basic results of the theory.

Applications of hypergroups appear in special subclasses like polygroups that they were studied by Comer [2], also see [4, 5, 6]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible in itself multigroup.

Definition 2.1 ([2])**.** A *polygroup* is a multi-valued system $\mathcal{M} = \langle P, \circ, e, ^{-1} \rangle$, with $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold for all x, y, z in P:

(1) $(x \circ y) \circ z = x \circ (y \circ z)$,

(2) $e \circ x = x \circ e = x$,

(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

In the above definition, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of P, and if $x \in P$ and A, B are non-empty subsets of P, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$. The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$ and $(x^{-1})^{-1} = x$. In the rest of this section we present the facts about polygroups that underlie the subsequent material. For further discussion of polygroups, we refer to Davvaz's book [4]. Many important examples of polygroups are collected in [4] such as Double coset algebra, Prenowitz algebras, Conjugacy class polygroups, Character polygroups, Extension of polygroups, and Chromatic polygroups. Clearly, every group is a polygroup. If K is a non-empty subset of P, then K is called a *subpolygroup* of P if $e \in K$ and $\langle K, \circ, e, ^{-1} \rangle$ is a polygroup. The subpolygroup N of P is said to be *normal* in P if $a^{-1} \circ N \circ a \subseteq N$, for every $a \in P$. If N is a normal subpolygroup of P, then $\langle P/N, \bullet, N, ^{-I} \rangle$ is a polygroup, where $N \circ a \bullet B \circ b = \{N \circ c \mid c \in N \circ a \circ b\}$ and $(N \circ a)^{-I} = N \circ a^{-1}$ [4]. There are several kinds of homomorphisms between polygroups [4]. In this paper, we apply only the following kinds of homomorphism.

Definition 2.2. Let $\langle P, \cdot, e, ^{-1} \rangle$ and $\langle P', \star, e, ^{-1} \rangle$ be two polygroups. Let ϕ be a mapping from P into P' such that $\phi(e) = e$. Then ϕ is called

(1) an *inclusion homomorphism* if $\phi(a \circ b) \subseteq \phi(a) \star \phi(b)$, for all $a, b \in P$,

(2) a *weak homomorphism* if $\phi(a \circ b) \cap \phi(a) \star \phi(b) \neq \emptyset$, for all $a, b \in P$,

(3) a *strong homomorphism* if $\phi(a \circ b) = \phi(a) \star \phi(b)$, for all $a, b \in P$.

A strong homomorphism ϕ is said to be an *isomorphism* if ϕ is one to one and onto. Two polygroups P and P' are said to be *isomorphic* if there is an isomorphism from P onto P'. The defining condition for any types of homomorphism is also valid for sets. For instance, if ϕ is a weak homomorphism of P into P' and A, B are nonempty subsets of P, then it follows that $f(A \circ B) \cap f(A) \star f(B) \neq \emptyset$.

By using the concept of generalized permutation, in [8], Davvaz defined permutation polygroups and action of a polygroup on a set. For the definition of crossed polymodule, we need the notion of polygroup action.

Definition 2.3 ([8])**.** Let $\mathcal{P} = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup and Ω be a non-empty set. A map $\alpha : P \times \Omega \to \mathcal{P}^*(\Omega)$, where $\alpha(g, \omega) := {}^{g}\omega$ is called a *(left) polygroup action* on Ω if the following axioms hold:

(1) ${}^{e}\omega = \omega$,

(2) ${}^{h}({}^{g}\omega) = {}^{h\circ g}\omega$, where ${}^{g}A = \bigcup\limits_{a\in A} {}^{g}a$ and ${}^{B}\omega = \bigcup\limits_{b\in B} {}^{b}\omega$, for all $A \subseteq \Omega$ and $B \subseteq P$,

(3) $\bigcup\limits_{\omega\in\Omega} {}^{g}\omega = \Omega$,

(4) for all $g \in P$, $a \in {}^{g}b \Rightarrow b \in {}^{g^{-1}}a$.

Example 2.4. Suppose that $\langle P, \circ, e, {}^{-1} \rangle$ is a polygroup. Then, P acts on itself by conjugation. Indeed, if we consider the map $\alpha : P \times P \to \mathcal{P}^*(P)$ by $\alpha(g, x) = {}^{g}x := g \circ x \circ g^{-1}$, then

(1) ${}^{e}x = x$,

(2)
$$\begin{aligned} {}^{h}({}^{g}x) &= {}^{h}(g \circ x \circ g^{-1}) = h \circ g \circ x \circ g^{-1} \circ h^{-1} \\ &= (h \circ g) \circ x \circ (h \circ g)^{-1} = \bigcup_{b\in h\circ g} (b \circ x \circ b^{-1}) \\ &= \bigcup_{b\in h\circ g} {}^{b}x = {}^{h\circ g}x, \end{aligned}$$

(3) $\bigcup\limits_{x\in P} {}^{g}x = \bigcup\limits_{x\in P} g \circ x \circ g^{-1} = P$,

(4) if $a \in {}^{g}b = g \circ b \circ g^{-1}$, then $g \in a \circ g \circ b^{-1}$ and hence $b^{-1} \in g^{-1} \circ a^{-1} \circ g$. This implies that $b \in g^{-1} \circ a \circ g$.

3. Crossed Polymodules as a Generalization of Crossed Modules

Now, in this section, we present the notion of crossed polymodule and main results about fundamental relation on polygroups and fundamental crossed polymodule.

Definition 3.1. A *crossed polymodule* $\mathcal{X} = (C, P, \partial, \alpha)$ consists of polygroups $\langle C, \star, e, {}^{-1}\rangle$ and $\langle P, \circ, e, {}^{-1} \rangle$ together with a strong homomorphism $\partial : C \to P$ and a (left) action $\alpha : P \times C \to \mathcal{P}^*(C)$ on C, satisfying the conditions:

(1) $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$, for all $c \in C$ and $p \in P$,

(2) ${}^{\partial(c)}c' = c \star c' \star c^{-1}$, for all $c, c' \in C$.

When we wish to emphasize the codomain P, we call $\mathcal{X}$ a *crossed P-polymodule.* The strong homomorphism $\partial : C \to P$ is called the *boundary* homomorphism.

Example 3.2. A conjugation crossed polymodule is an inclusion of a normal subpolygroup N of P, with action given by conjugation. In particular, for any polygroup

P the identity map $Id_P : P \to P$ is a crossed polymodule with the action of P on itself by conjugation. Indeed, there are two canonical ways in which a polygroup P may be regarded as a crossed polymodule: via the identity map or via the inclusion of the trivial subpolygroup.

Example 3.3. If C is a P-polymodule, then there is a well defined action α of P on C. This together with the zero homomorphism yields a crossed polymodule $(C, P, 0, \alpha)$.

Example 3.4. The direct product of $\mathcal{X}_1 \times \mathcal{X}_2$ of two crossed polymodules has source $C_1 \times C_2$, range $P_1 \times P_2$ and boundary homomorphism $\partial_1 \times \partial_2$ with $P_1 \times P_2$ acting trivially on $C_1 \times C_2$.

Every crossed module is a crossed polymodule.

Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. We define the relation β^*_P as the smallest equivalence relation on P such that the quotient P/β^*_P, the set of all equivalence classes, is a group. In this case β^*_P is called the *fundamental equivalence relation* on P and P/β^*_P is called the *fundamental group*. The product $\odot$ in P/β^*_P is defined as follows: $\beta^*_P(x) \odot \beta^*_P(y) = \beta^*_P(z)$, for all $z \in \beta^*_P(x) \circ \beta^*(y)$. This relation is introduced by Koskas [13] and studied mainly by Corsini [3], Leoreanu-Fotea [14] and Freni [10, 11] concerning hypergroups, Vougiouklis [17] concerning H_v-groups, Davvaz concerning polygroups [7], and many others. We consider the relation β_P as follows:

$$x \ \beta_P \ y \ \Leftrightarrow \ \text{there exist } z_1, \ldots z_n \text{ such that } \{x, y\} \subseteq \circ \prod_{i=1}^{n} z_i = z_1 \circ \ldots \circ z_n.$$

Freni in [10] proved that for hypergroups $\beta = \beta^*$. Since polygroups are certain subclass of hypergroups, we have $\beta^*_P = \beta_P$. The kernel of the *canonical map* $\varphi_P : P \longrightarrow P/\beta^*_P$ is called the *core* of P and is denoted by ω_P. Here we also denote by ω_P the unit of P/β^*_P. It is easy to prove that the following statements: $\omega_P = \beta^*_P(e)$ and $\beta^*_P(x)^{-1} = \beta^*_P(x^{-1})$, for all $x \in P$.

Throughout the paper, we denote the binary operations of the fundamental groups P/β^*_P and C/β^*_C by $\odot$ and $\otimes$, respectively.

Now, we can consider another notion of the kernel of a strong homomorphism of polygroups. Let $\langle P, \circ, e, ^{-1} \rangle$ and $\langle C, \star, e, ^{-1} \rangle$ be two polygroups and $\partial : C \to P$ be a strong homomorphism. The *core-kernel* of ∂ is defined by

$$ker^* \partial = \{x \in C \mid \partial(x) \in \omega_P\}.$$

Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule. Then, $ker^*\partial$ is a $P/\partial(C)$-polymodule [9].

Proposition 3.5 ([9])**.** *Let $\langle C, \star, e, ^{-1}\rangle$ and $\langle P, \circ, e, ^{-1}\rangle$ be two polygroups and let $\partial : C \to P$ be a strong homomorphism. Then, ∂ induces a group homomorphism $\mathcal{D} : C/\beta^*_C \to P/\beta^*_P$ by setting*

$$\mathcal{D}(\beta^*_C(c)) = \beta^*_P(\partial(c)), \text{ forall } c \in C.$$

We say the action of P on C is *productive*, if for all $c \in C$ and $p \in P$ there exist $c_1, \ldots, c_n$ in C such that $c^p = c_1 \star \ldots \star c_n$.

Example 3.6. The action defined in Example 2.4 is productive.

Let $\langle C, \star, e, ^{-1}\rangle$ and $\langle P, \circ, e, ^{-1}\rangle$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. We define the map $\psi : P/\beta^*_P \times P/\beta^*_C \to \mathcal{P}^*(P/\beta^*_C)$ as usual manner:

$$\psi(\beta^*_P(p), \beta^*_C(c)) = \{\beta^*_C(x) \mid x \in \bigcup_{\substack{y \in \beta^*_C(c)\\ z \in \beta^*_P(p)}} {}^{z}y\}.$$

By definition of β^*_C, since the action of P on C is productive, we conclude that $\psi(\beta^*_P(p), \beta^*_C(c)$ is singleton, i.e., we have

$$\psi : P/\beta^*_P \times P/\beta^*_C \to P/\beta^*_C,$$
$$\psi(\beta^*_P(p), \beta^*_C(c)) = \beta^*_C(x), \text{ for all } x \in \bigcup_{\substack{y \in \beta^*_C(c)\\ z \in \beta^*_P(p)}} {}^{z}y.$$

We denote $\psi(\beta^*_P(p), \beta^*_C(c)) = {}^{[\beta^*_P(p)]}[\beta^*_C(c)]$.

Proposition 3.7 ([9])**.** *Let $\langle C, \star, e, ^{-1}\rangle$ and $\langle P, \circ, e, ^{-1}\rangle$ be two polygroups and let $\alpha : P \times C \to \mathcal{P}^*(C)$ be a productive action on C. Then, ψ is an action of the group P/β^*_P on the group P/β^*_C.*

Theorem 3.8 ([1])**.** *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crosed polymodule such that the action of P on C is productive. Then, $\mathcal{X}_{\beta^*} = (C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$ is a crossed module.*

4. Derivation of Crossed Modules

Whitehead showed in [18] that the set of derivation of a crossed module has a natural module structure and he characterized its group of units. In this section we introduce the notion of derivation of a crossed polymodule and actor crossed polymodules by using Norrie's way.

Definition 4.1. Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule. A *derivation* $\eta : P \to C$ is a function satisfying

$$\eta(x \circ y) = \eta(x) \star \ {}^{x}\eta(y),$$

for all $x, y \in P$.

Lemma 4.2. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule. If $\eta : P \to C$ is a derivation, then*

(1) $\eta(e) = e$,
(2) ${}^{x}\eta(x^{-1}) = \eta(x)^{-1}$.

Proof. The proof of (1) is clear. We prove (2). Since $e \in x \circ x^{-1}$, $\eta(e) \in \eta(x \circ x^{-1}$. By (1), we conclude that $e \in \eta(x \circ x^{-1})$. Thus, $e \in \eta(x) \star \ {}^{x}\eta(x^{-1})$. Now, by third condition of definition of polygroup, we obtain ${}^{x}\eta(x^{-1}) \in \eta(x)^{-1} \star e$. Therefore, ${}^{x}\eta(x^{-1}) = \eta(x)^{-1}$. □

Theorem 4.3. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule and $\eta : P \to C$ be a derivation. Then, the following map*

$$\eta^* : P/\beta^*_P \to C/\beta^*_C$$
$$\eta^* \left(\beta^*_P(p)\right) = \beta^*_C(\eta(p))$$

is a derivation for crossed module $(C/\beta^*_C, P/\beta^*_P, \mathcal{D}, \psi)$.

Proof. Suppose that $x, y \in P$ are arbitrary. Then, we have

$$\begin{aligned}
\eta^* \left(\beta^*_P(x) \odot \beta^*_P(y)\right)) &= \eta^* \left(\beta^*_P(x \circ y)\right) \\
&= \beta^*_C(\eta(x \circ y)) \\
&= \beta^*_C \left(\eta(x) \star \ {}^{x}\eta(y)\right) \\
&= \beta^*_C(\eta(x)) \otimes \beta^*_C \left({}^{x}\eta(y)\right) \\
&= \beta^*_C(\eta(x)) \otimes \ {}^{\left[\beta^*_P(x)\right]} \left[\beta^*_C(\eta(y))\right] \\
&= \beta^* \left(\beta^*_P(x)\right) \otimes \ {}^{\left[\beta^*_P(x)\right]} \left[\eta^* \left(\beta^*_P(y)\right)\right].
\end{aligned}$$

□

In the following proposition, $\mathcal{P}^*(P)$ and $\mathcal{P}^*(C)$ are semihypergroups. So, we can consider ρ and σ as weak homomorphisms between semihypergroups.

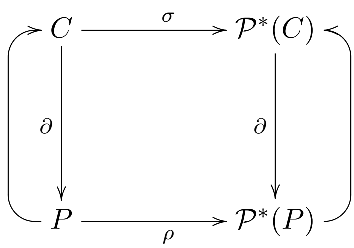
Proposition 4.4. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule and $\eta : P \to C$ be a derivation. Then, η defines weak homomorphisms ρ and σ, where*

(1) $\rho : P \to \mathcal{P}^*(P)$, such that $\rho(p) = \partial(\eta(p)) \circ p$, for all $p \in P$,
(2) $\sigma : C \to \mathcal{P}^*(C)$, such that $\sigma(c) = \eta(\partial(c)) \star c$, for all $c \in C$.

Moreover,

(i) $\rho\partial(c) = \partial\sigma(c)$
(ii) $\sigma\eta(p) \subseteq \eta\rho(p)$
(iii) $\sigma(\ {}^{p}c) \supseteq\ {}^{\rho(p)}(\sigma(c))$

for all $c \in C$ and $p \in P$.



Proof. (1) By definition, $\rho(e) = \partial(\eta(e)) \circ e$. By Lemma 4.2, we obtain $\rho(e) = \partial(e) = e$. Now, suppose that $p_1, p_2 \in P$ are arbitrary. Then, we have

$$\begin{aligned}\rho(p_1 \circ p_2) &= \{\rho(p) \mid p \in p_1 \circ p_2\}\\ &= \{\partial(\eta(p)) \circ p \mid p \in p_1 \circ p_2\}\\ &= \partial(\eta(p_1 \circ p_2)) \circ p_1 \circ p_2\\ &= \partial\left(\eta(p_1) \star\ {}^{p_1}\eta(p_2)\right) \circ p_1 \circ p_2\\ &= \partial(\eta(p_1)) \circ \partial\left(\ {}^{p_1}\eta(p_2)\right) \circ p_1 \circ p_2\\ &= \partial(\eta(p_1)) \circ p_1 \circ \partial(\eta(p_2)) \circ p_1^{-1} \circ p_1 \circ p_2.\end{aligned}$$

Since $e \in p_1^{-1} \circ p_1$, we conclude that

$$\rho(p_1 \circ p_2) \cap \Big(\partial(\eta(p_1)) \circ p_1 \circ \partial(\eta(p_2)) \circ p_2\Big) \neq \emptyset.$$

Hence, $\rho(p_1 \circ p_2) \cap \rho(p_1) \circ \rho(p_2) \neq \emptyset$. Therefore, ρ is a weak endomorphism.

(2) Clearly, $\sigma(e) = \eta(\partial(e)) \star e = \eta(\partial(e)) = \eta(e) = e$. Now, suppose that $c_1, c_2 \in C$ are arbitrary. Then, we have

$$\begin{aligned}\sigma(c_1 \star c_2) &= \{\sigma(c) \mid c \in c_1 \star c_2\}\\ &= \{\eta(\partial(c)) \star c \mid c \in c_1 \star c_2\}\end{aligned}$$

$$
\begin{aligned}
&= \eta(\partial(c_1 \star c_2)) \star c_1 \star c_2 \\
&= \eta(\partial(c_1) \circ \partial(c_2)) \star c_1 \star c_2 \\
&= \eta(\partial(c_1)) \star^{\partial(c_1)} \eta(\partial(c_2)) \star c_1 \star c_2 \\
&= \eta(\partial(c_1)) \star c_1 \star \eta(\partial(c_2)) \star c_1^{-1} \star c_1 \star c_2
\end{aligned}
$$

Since $e \in c_1^{-1} \star c_1$, we conclude that

$$\sigma(c_1 \star c_2) \cap \Big(\eta(\partial(c_1)) \star c_1 \star \eta(\partial(c_2)) \star c_2\Big) \neq \emptyset.$$

Hence, $\sigma(c_1 \star c_2) \cap \sigma(c_1) \star \sigma(c_2) \neq \emptyset$. Therefore, σ is a weak endomorphism.

Now, we prove (i), (ii) and (iii).

(i)

$$
\begin{aligned}
\rho\partial(c) &= \partial(\eta(\partial(c))) \circ \partial(c) \\
&= \partial(\eta(\partial(c)) \star c) \\
&= \partial\sigma(c).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\sigma\eta(p) &= \eta(\partial(\eta(p))) \star \eta(p) \\
&= \eta(\partial(\eta(p))) \star \eta(p) \star e \\
&\subseteq \eta(\partial(\eta(p))) \star \eta(p) \star \eta(p) \star \eta(p)^{-1} \\
&= \eta(\partial(\eta(p))) \star {}^{\partial(\eta(p))}\eta(p) \\
&= \eta(\partial(\eta(p)) \circ p) \\
&= \eta\rho(p).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\sigma({}^{p}c) &= \eta(\partial({}^{p}c)) \star {}^{p}c \\
&= \eta(p \circ \partial(c) \circ p^{-1}) \star {}^{p}c \\
&= \eta(p) \star {}^{p}(\eta(\partial(c))) \star {}^{p\circ\partial(c)}(\eta(p^{-1})) \star {}^{p}c \\
&= \eta(p) \star {}^{p}(\eta(\partial(c))) \star {}^{p\circ\partial(c)\circ p^{-1}}(\eta(p^{-1})) \star {}^{p}c \\
&= \eta(p) \star {}^{p}(\eta(\partial(c))) \star {}^{\partial({}^{p}c)}(\eta(p^{-1})) \star {}^{p}c \\
&= \eta(p) \star {}^{p}(\eta(\partial(c))) \star \{x \star \eta(p)^{-1} \star x^{-1} \mid x \in {}^{p}c\} \star {}^{p}c \\
&\supseteq \eta(p) \star {}^{p}(\eta(\partial(c))) \star {}^{p}c \star \eta(p)^{-1} \\
&= \eta(p) \star {}^{p}(\eta(\partial(c)) \star c) \star \eta(p)^{-1}
\end{aligned}
$$

$$= \eta(p) \star {}^{p}\sigma(c) \star \eta(p)^{-1}$$
$$= {}^{\partial(\eta(p))}({}^{p}(\sigma(c)))$$
$$= {}^{\partial(\eta(p))\circ p}\sigma(c)$$
$$= {}^{\rho(p)}\sigma(c).$$

□

We denote by $\mathrm{Der}(\mathcal{X})$ the set of all derivations from P to C and $\mathrm{Der}(\mathcal{X}/\beta^*)$ the set of all derivations from $P/_{\beta_P^*}$ to $C/_{\beta_C^*}$. By using the previous proposition, we obtain Whitehead multiplication in $\mathrm{Der}(\mathcal{X}/_{\beta^*})$ as follows:

Corollary 4.5. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule and $\eta : P \to C$ be a derivation. Then, η defines endomorphisms ρ^* and σ^* of groups, where*

(1) $\rho^* : P/_{\beta_P^*} \to P/_{\beta_P^*}$, *such that* $\rho^*(\beta_P^*(p)) = \mathcal{D}(\eta^*(\beta_P^*(p))) \odot \beta_P^*(p)$, *for all* $p \in P$,

(2) $\sigma^* : C/_{\beta_C^*} \to C/_{\beta_C^*}$, *such that* $\sigma^*(\beta_C^*(c)) = \eta^*(\mathcal{D}(\beta_C^*(c))) \otimes \beta_C^*(c)$, *for all* $c \in C$.

Moreover,

(i) $\rho^*\mathcal{D}(\beta_C^*(c)) = \mathcal{D}\sigma^*(\beta_C^*(c))$,

(ii) $\sigma^*\eta^*(\beta_P^*(p)) = \eta^*\rho^*(\beta_P^*(p))$,

(iii) $\sigma^*\left({}^{[\beta_P^*(p)]}[\beta_C^*(c)]\right) = {}^{[\sigma^*(\beta_P^*(p))]}[\sigma^*(\beta_C^*(c))]$, *for all* $c \in C$ *and* $p \in P$.

The Whitehead multiplication in $\mathrm{Der}(\mathcal{X}/\beta^*)$ is defined as follows:

$$(\eta_1^* \cdot \eta_2^*)(\beta_P^*(p)) = (\eta_1^*\mathcal{D}\eta_2^*)(\beta_P^*(p)) \otimes \eta_2^*(\beta_P^*(p)) \otimes \eta_1^*(\beta_P^*(p)).$$

Then, $\eta_1^* \cdot \eta_2^* \in \mathrm{Der}(\mathcal{X}/\beta^*)$ and this multiplication is associative. Therefore, this multiplication turns $\mathrm{Der}(\mathcal{X}/\beta^*)$ into a monoid. If we denote ρ^* $(= \rho_{\eta^*}^*)$ and σ^* $(= \sigma_{\eta^*}^*)$, then we have

$$\rho^*_{\eta_1^* \cdot \eta_2^*} = \rho^*_{\eta_1^*}\rho^*_{\eta_2^*} \quad \text{and} \quad \sigma^*_{\eta_1^* \cdot \eta_2} = \sigma^*_{\eta_1^*}\sigma^*_{\eta_2^*}$$

The Whitehead group $\mathcal{W}(\mathcal{X}/\beta^*)$ is defined to be the group of units of $\mathrm{Der}(\mathcal{X}/\beta^*)$. The following corollary give us the results of [19] about the fundamental crossed module derived from a crossed polymodule.

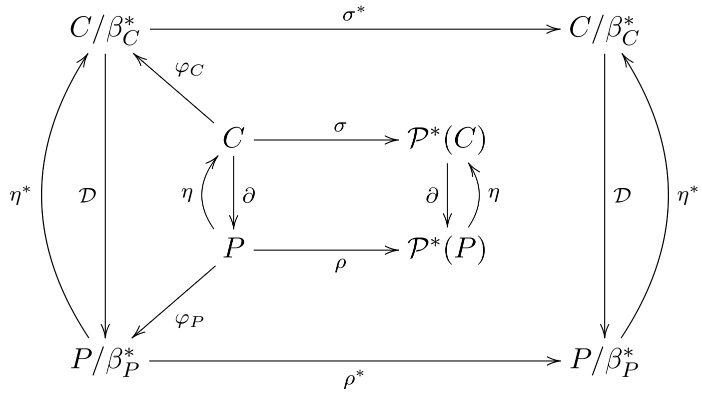
Corollary 4.6. *The following conditions on $\eta \in Der(\mathcal{X})$ are equivalent.*

(1) $\eta^* \in \mathcal{W}(\mathcal{X}/\beta^*)$,

(2) $\rho^* : P/_{\beta_P^*} \to P/_{\beta_P^*}$ *is an automorphism,*

(3) $\sigma^* : C/_{\beta_C^*} \to C/_{\beta_C^*}$ *is an automorphism.*

Corollary 4.7. *The following diagram is commutative.*



Corollary 4.8. *We have*

(1) $\varphi_P\rho = \rho^*\varphi_P$,

(2) $\varphi_C\sigma = \sigma^*\varphi_C$.

Proof. (1) For every $p \in P$, we have

$$\begin{aligned}
\varphi_P\rho(p) &= \beta_P^*(\rho(p))\\
&= \beta_P^*(\partial(\eta(p)) \circ p)\\
&= \beta_P^*(\partial(\eta(p))) \odot \beta_P^*(p)\\
&= \mathcal{D}(\beta_C^*(\eta(p))) \odot \beta_P^*(p)\\
&= \mathcal{D}(\eta^*(\beta_P^*(p))) \odot \beta_P^*(p)\\
&= \rho^*(\beta_P^*(p))\\
&= \rho^*\varphi_P(p).
\end{aligned}$$

(2) For every $c \in C$, we have

$$\begin{aligned}
\varphi_C\sigma(c) &= \varphi_C(\eta(\partial(c)) * c)\\
&= \beta_C^*(\eta(\partial(c)) * c)\\
&= \beta_C^*(\eta(\partial(c))) \otimes \beta_C^*(c)\\
&= \eta^*(\beta_P^*(\partial(c))) \otimes \beta_C^*(c)\\
&= \eta^*\mathcal{D}(\beta_C^*(c)) \otimes \beta_C^*(c)\\
&= \sigma^*(\beta_C^*(c))\\
&= \sigma^*\varphi_C(c).
\end{aligned}$$

□

Let $\mathcal{X} = (P, C, \partial, \alpha)$ be a crossed polymodule. We set

$$\mathrm{Der}_{\beta^*}(\mathcal{X}) = \{\eta : \mathrm{P} \to \mathrm{C} \ \mid \ \eta^* \in \mathrm{Der}(\mathcal{X}_{\beta^*})\}.$$

Then, we define the following hyperoperation on $\mathrm{Der}_{\beta^*}(\mathcal{X})$

$$\eta_1 \blacktriangledown \eta_2 = \{\eta \ \mid \ \eta(p) \in (\eta_1 \partial \eta_2)(p) \circ \eta_2(p) \circ \eta_1(p), \text{for all p} \in \mathrm{P}\}.$$

Theorem 4.9. *The above hyperoperation is well-defined.*

Proof. Indeed, we must show that $\eta_1 \blacktriangledown \eta_2 \subseteq \mathrm{Der}_{\beta^*}(\mathcal{X})$. Suppose that $\eta \in \eta_1 \blacktriangledown \eta_2$. Then, for all $p \in P$, we have

$$\eta(p) \in (\eta_1 \partial \eta_2)(p) \star \eta_2(p) \star \eta_1(p).$$

Thus,

$$\beta_C^*(\eta(p)) = \beta_C^*(\eta_1 \partial \eta_2(p)) \otimes \beta_C^*(\eta_2(p)) \otimes \beta_C^*(\eta_1(p)),$$

for all $p \in P$. This implies that

$$\eta^*(\beta_P^*(p)) = \eta_1^* \mathcal{D} \eta_2^*(\beta_P^*(p)) \otimes \eta_2^*(\beta_P^*(p)) \otimes \eta_1^*(\beta_P^*(p)).$$

For simplify we take $\beta_P^*(p) = v$. Then,

$$\eta^*(v) = \eta_1^* \mathcal{D} \eta_2^*(v) \otimes \eta_2^*(v) \otimes \eta_1^*(v). \qquad (I)$$

Now, by following Whitehead [19], the above relation give us a derivation. □

Corollary 4.10. *For all $\eta, \eta' \in \eta_1 \blacktriangledown \eta_2$, we have $\eta^* = \eta'^*$.*

Proof. It is obviose by the relation (I). □

Theorem 4.11. $(\mathrm{Der}_{\beta^*}(\mathcal{X}), \blacktriangledown)$ *is a semihypergroup with identity.*

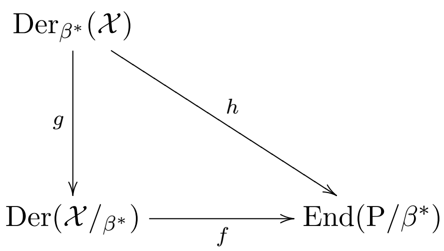
Proof. Associativity law holds. Because

$$\begin{aligned}
&\eta_1 \blacktriangledown (\eta_2 \blacktriangledown \eta_3) \\
&= \eta_1 \blacktriangledown \{\eta \ \mid \ \eta \in \eta_2 \blacktriangledown \eta_3\} \\
&= \eta_1 \blacktriangledown \{\eta' \ \mid \ \eta'(p) \in (\eta_2 \partial \eta_3(p)) \circ \eta_3(p) \circ \eta_2(p)\} \\
&= \{\eta \ \mid \ \eta(p) \in (\eta_1 \partial \eta')(p) \circ \eta'(p) \circ \eta_1(p), \eta'(p) \in (\eta_2 \partial \eta_3)(p) \circ \eta_3(p) \circ \eta_2(p)\} \\
&= \{\eta \ \mid \ \eta(p) \in (\eta_1 \partial [\eta_2 \partial \eta_3(p) \circ \eta_3(p) \circ \eta_2(p)]) \circ (\eta_2 \partial \eta_3)(p) \circ \eta_3(p) \\
&\qquad \circ \eta_2(p) \circ \eta_1(p)\}
\end{aligned}$$

$$
\begin{aligned}
&= \{\eta \mid \eta(p) \in \eta_1[\partial\eta_2\partial\eta_3(p) \circ \partial(\eta_3(p)) \circ \partial(\eta_2(p))] \circ (\eta_2\partial\eta_3(p)) \circ \eta_3(p) \\
&\qquad \circ \eta_2(p) \circ \eta_1(p)\} \\
&= \{\eta \mid \eta(p) \in \eta_1(\partial\eta_2\partial\eta_3(p) \circ \partial(\eta_3(p))) \circ {}^{(\partial\eta_2\partial\eta_3(p)\circ\partial(\eta_3(p)))}\eta_1(\partial\eta_2(p)) \\
&\qquad \circ \eta_2\partial\eta_3(p) \circ \eta_3(p) \circ \eta_2(p) \circ \eta_1(p)\} \\
&= \{\eta \mid \eta(p) \in \eta_1(\partial\eta_2\partial\eta_3(p)) \circ {}^{(\partial\eta_2\partial\eta_3(p))}\eta_1(\partial(\eta_3(p))) \\
&\qquad \circ {}^{(\partial\eta_2\partial\eta_3(p)\circ\partial(\eta_3(p)))}\eta_1(\partial\eta_2(p)) \circ (\eta_2\partial\eta_3(p)) \circ \eta_3(p) \circ \eta_2(p) \circ \eta_1(p)\} \\
&= \{\eta \mid \eta(p) \in (\eta_1\partial\eta_2\partial\eta_3(p)) \circ (\eta_2\partial\eta_3(p)) \circ (\eta_1\partial\eta_3(p)) \circ ((\eta_2\partial\eta_3)(p))^{-1} \\
&\qquad \circ (\eta_2(\partial(\eta_3(p)))) \circ (\eta_1(\partial\eta_2(p))) \circ \eta_3(p) \circ (\eta_3(p))^{-1} \\
&\qquad \circ ((\eta_2\partial\eta_3)(p))^{-1}(\eta_2\partial\eta_3(p)) \circ \eta_3(p) \circ \eta_2(p) \circ \eta_1(p)\} \\
&= \{\eta \mid \eta(p) \in (\eta_1\partial\eta_2\partial\eta_3(p)) \circ (\eta_2\partial\eta_3(p)) \circ (\eta_1\partial\eta_3(p)) \circ \eta_3(p) \circ (\eta_1\partial\eta_2(p)) \\
&\qquad \circ \eta_2(p) \circ \eta_1(p)\} \\
&= \{\eta \mid \eta'(p) \in (\eta_1\partial\eta_2(p)) \circ \eta_2(p) \circ \eta_1(p), \eta(p) \in (\eta'\partial\eta_3(p)) \circ \eta_3(p) \circ \eta'(p)\} \\
&= \{\eta \mid \eta'(p) \in (\eta_1\partial\eta_2(p)) \circ \eta_2(p) \circ \eta_1(p)\} \blacktriangledown \eta_3 \\
&= (\eta_1 \blacktriangledown \eta_2) \blacktriangledown \eta_3
\end{aligned}
$$

So, $(\mathrm{Der}_{\beta^*}(\mathcal{X}), \blacktriangledown)$ is a semihypergroup. The element which maps each element of P into the identity element of C, is the identity element of $\mathrm{Der}_{\beta^*}(\mathcal{X})$. □

Corollary 4.12. *The following diagram is a commutative diagram of semihypergroups with identity.*



Note that every semigroup can consider as a semihypergroup.

The polygroup $\mathcal{DA}(\mathcal{X})$ is defined to be the polygroup of invertible and reversible elements of $Der_{\beta^*}(\mathcal{X})$.

Proposition 4.13. *There is a homomorphism $f : \mathcal{DA}(\mathcal{X}) \to \mathcal{W}(\mathcal{X}/\beta^*)$ by $\eta \mapsto \eta^*$.*

Proof. We have $f(\eta_1 \blacktriangledown \eta_2) = \{f(\eta) \mid \eta \in \eta_1 \blacktriangledown \eta_2\} = \{\eta^* \mid \eta \in \eta_1 \blacktriangledown \eta_2\}$. Since $\eta_1^* \cdot \eta_2^* \in \eta_1 \blacktriangledown \eta_2$, by Corollary 4.10, we conclude that $f(\eta_1 \blacktriangledown \eta_2) = \eta_1^* \cdot \eta_2^* = f(\eta_1) \cdot f(\eta_2)$. □

Proposition 4.14. *There is a homomorphism* $g : \mathcal{W}(\mathcal{X}/\beta^*) \to Aut(\mathcal{X}/\beta^*)$ *by* $\eta^* \mapsto (\sigma^*_{\eta^*}, \rho^*_{\eta^*})$.

Proof. By Corollaries 4.5 and 4.6, it is obviouse. □

Corollary 4.15. *There is the homomorphism* $gf : \mathcal{DA}(\mathcal{X}) \to Aut(\mathcal{X}/\beta^*)$.

An action of $Aut(\mathcal{X}/\beta^*)$ on the group $\mathcal{W}(\mathcal{X}/\beta^*)$ is defined by

$$T((u,v),\eta^*) = {}^{(u,v)}\eta^* = u^{-1}\eta^* v.$$

Theorem 4.16. *Let* $\mathcal{X} = (C, P, \partial, \alpha)$ *be a crossed polymodule. Then,*

$$\mathcal{W}(\mathcal{X}/\beta^*), Aut(\mathcal{X}/\beta^*), g, T)$$

is a crossed module.

Proof. The proof is similar to the results of Norrie about crossed modules [16]. □

Lemma 4.17. *For every* $(u, v) \in Aut(\mathcal{X}/\beta^*)$, *we have*

(1) $\rho^*_{u^{-1}\eta^* v} = v{-}1\rho^*_{\eta^*} v$,
(2) $\sigma^*_{u^{-1}\eta^* v} = u^{-1}\sigma^*_{\eta^*} u$.

Proof. (1) We have

$$\begin{aligned}
\rho^*_{u^{-1}\eta^* v}(\beta^*_P(p)) &= \mathcal{D}u^{-1}\eta^* v(\beta^*_P(p)) \odot \beta^*_P(p)\\
&= v^{-1}\mathcal{D}\eta^* v(\beta^*_P(p)) \odot \beta^*_P(p)\\
&= v^{-1}\left(\mathcal{D}\eta^* v(\beta^*_P(p)) \odot v(\beta^*_P(p))\right)\\
&= v^{-1}\rho^*_{\eta^*} v(\beta^*_P(p)).
\end{aligned}$$

(2) We have

$$\begin{aligned}
\sigma^*_{u^{-1}\eta^* v}(\beta^*_C(c)) &= u^{-1}\eta^* v(\mathcal{D}(\beta^*_C(c)) \otimes \beta^*_C(c)\\
&= u^{-1}\eta^*\mathcal{D}u(\beta^*_C(c)) \otimes \beta^*_C(c)\\
&= u^{-1}\left(\eta^*\mathcal{D}u(\beta^*_C(c)) \otimes u(\beta^*_C(c))\right)\\
&= u^{-1}\sigma^*_{\eta^*} u(\beta^*_C(c)).
\end{aligned}$$

□

Now, we can consider another action of $Aut(\mathcal{X}/\beta^*)$ on the polygroup $\mathcal{DA}(\mathcal{X})$ by

$$\begin{aligned}
&\widehat{T} : Aut(\mathcal{X}/\beta^*) \times \mathcal{DA}(\mathcal{X}) \to \mathcal{P}^*(\mathcal{DA}(\mathcal{X}))\\
&\widehat{T}((u,v)),\eta) = \{\eta' \mid {\eta'}^* = u^{-1}\eta^* v\}.
\end{aligned}$$

Indeed, ${}^{(u,v)}\eta = \{\eta' \mid {\eta'}^{*} = u^{-1}\eta^{*}v\}$.

Theorem 4.18. *Let $\mathcal{X} = (C, P, \partial, \alpha)$ be a crossed polymodule. Then,*

$$(\mathcal{DA}(\mathcal{X}), Aut(\mathcal{X}/\beta^{*}), gf, \widehat{T})$$

is a crossed polymodule.

Proof. By Lemma 4.17, it is straightforward. □

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[a]Department of Mathematics, Yazd University, Yazd, Iran
Email address: davvaz@yazd.ac.ir

[b]Department of Mathematics, American University of the Middle East, Kuwait
Email address: murat.alp@aum.edu.kw