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GENERALIZATION OF SOME INEQUALITIES TO THE CLASS OF GENERALIZED DERIVATIVE

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Abstract. In this paper, we obtain some inequalities concerning the class of generalized derivative and generalized polar derivative which are analogous respectively to the ordinary derivative and polar derivative of polynomials.

1. Introduction

Let $\mathcal{P}_n$ denote the space of all algebraic polynomials of the form $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ of degree n and let $P'(z)$ be its derivative. Let R_+^n be the set of all n tuples $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of positive real numbers with $\gamma_k \geq 1, \quad 1 \leq k \leq n$, the polynomial $G_\gamma[P](z) = \sum\limits_{k=1}^{n} \gamma_k \prod\limits_{v=1, v\neq k}^{n} (z - z_v)$ be a generalized derivative [8] of $P(z)$, where $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$. For $\gamma = (1, 1, \ldots, 1)$, the generalized derivative $G_\gamma[P](z)$ reduces to $P'(z)$. In this paper, we extend the some well-known results concerning the polynomials due to Dubinin [4], N.A. Rather et al. [7] to the class of generalized derivative.

Concerning the estimation of the lower bound of $Re\left(\frac{zP'(z)}{P(z)}\right)$ on $|z| = 1$, Dubinin [4] proved the following result.

Theorem 1.1. *If $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$*

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{2}\left(1 + \frac{1}{n}\left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right). \tag{1.1}$$

As a generalization of Theorem 1.1, N.A. Rather et al [7] have proved the following result.

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Theorem 1.2. *If $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$*

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right). \tag{1.2}$$

Also concerning the maximum of $|P'(z)|$ in terms of maximum of $|P(z)|$ on $|z| = 1$, Turán [9] showed that if $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Equality in inequality (1.3) holds for those polynomials $P \in \mathcal{P}_n$ which have all their zeros on $|z| = 1$.

Dubinin [4] used the boundary Schwarz lemma due to Osserman [6] to obtain an interesting refinement of (1.3), in fact, proved that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2}\left(1 + \frac{1}{n}\left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right) \max_{|z|=1} |P(z)|. \tag{1.4}$$

As a generalization of inequality (1.3), Malik [5] proved the following result.

Theorem 1.3. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

The result is sharp and best possible for $P(z) = (z + k)^n$. It is natural to look for a similar refinement of inequality (1.5), as the Dubinin [4] refined the inequality (1.3). In this direction N.A. Rather et al. [7] proved the result which is refinement of inequality (1.5) and generalization of inequality (1.4). In fact, they proved

Theorem 1.4. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s} z^{n-s}) \in \mathcal{P}_n$, $0 \leq s \leq n$, has all its zeros in $|z| \leq k$, $k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right) \max_{|z|=1} |P(z)|. \tag{1.6}$$

The result is sharp and equality holds for $P(z) = z^s(z+k)^{n-s}$, $s < n$. On setting $s = 0$ in Theorem 1.4, they proved the following corollary.

Corollary 1.5. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right) \max_{|z|=1} |P(z)|. \tag{1.7}$$

The result is sharp and equality holds for $P(z) = (z + k)^n$.

The polar derivative $D_\alpha P(z)$ of $P \in \mathcal{P}_n$ with respect to the point $\alpha \in \mathbb{C}$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z). \tag{1.8}$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z) \tag{1.9}$$

uniformly for $|z| \le R$, $R > 0$.

Definition 1.6. Define $N_\alpha^\gamma[P](z) = \sum_{k=1}^{n} \gamma_k P(z) + (\alpha - z)G_\gamma[P](z)$ with $\gamma \in R_+^n$ as the *generalized polar derivative* of $P(z)$.

The polynomial $N_\alpha^\gamma[P](z)$ generalized the polar derivative in the sense for $\gamma = (1, 1, \dots, 1)$, then $N_\alpha^\gamma[P](z)$ reduces to $D_\alpha P(z)$. In this paper we extend some results to class of generalized polar derivative of $P(z)$. A. Aziz [1] and Aziz and Rather [2] obtained several estimates for maximum modulus of $D_\alpha P(z)$ on $|z| = 1$ and among other thing, they [3] extended inequality (1.5) to the polar derivative. In fact, they proved

Theorem 1.7. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \le k \le 1$, then for $\alpha \in C$ with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n\frac{|\alpha| - k}{1 + k} \max_{|z|=1} |P(z)| \tag{1.10}$$

The result is sharp and best possible for $P(z) = (z + k)^n$. with $|\alpha| \ge k$.

Next N.A. Rather et al. [7] establish the following refinement of inequality (1.10). In fact, they proved

Theorem 1.8. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \le s \le n$, has all its zeros in $|z| \le k$, $k \le 1$, then for $\alpha \in C$ with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n\frac{|\alpha| - k}{1 + k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right) \max_{|z|=1} |P(z)|. \tag{1.11}$$

The result is sharp and equality holds for $P(z) = z^s(z + k)^{n-s}$, $s < n$, with $|\alpha| \ge k$. On setting $s = 0$ in Theorem 1.8, they proved the following corollary.

Corollary 1.9. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq n\frac{|\alpha| - k}{1+k}\left(1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right)\max_{|z|=1}|P(z)|. \tag{1.12}$$

The result is sharp and equality holds for $P(z) = (z+k)^n$, with $|\alpha| \geq k$.

2. Main Results

In this paper, we extend Theorem 1.2 and Theorem 1.4 to the class of generalized derivative. We get our first result which generalizes Theorem 1.2 to the *generalized derivative.* In fact, we prove

Theorem 2.1. *If $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$ and $\gamma \in R^n_+$ with $\gamma_j \geq 1, 1 \leq j \leq n$*

$$Re\left(\frac{zG_\gamma[P](z)}{P(z)}\right) \geq \frac{\sum_{j=1}^n \gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^n \gamma_j}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right). \tag{2.1}$$

On setting $k = 1$, we obtain the following result which is a generalization of Theorem 1.1.

Corollary 2.2. *If $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$ and $\gamma \in R^n_+$ with $\gamma_j \geq 1, 1 \leq j \leq n$*

$$Re\left(\frac{zG_\gamma[P](z)}{P(z)}\right) \geq \frac{\sum_{j=1}^n \gamma_j}{2}\left(1 + \frac{1}{\sum_{j=1}^n \gamma_j}\left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right). \tag{2.2}$$

Remark 2.3. For a n-tuple $\gamma = (1, 1, \ldots, 1)$, Theorem 2.1 and Corollary 2.2 reduce to Theorem 1.2 and Theorem 1.1.

As an application of Theorem 2.1, we present the following result which extends Theorem 2.4 to the class of generalized derivative. In fact, we prove

Theorem 2.4. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \leq s \leq n$, has all its zeros in $|z| \leq k$, $k \leq 1$, and $\gamma \in R^n_+$ with $\gamma_j \geq 1, 1 \leq j \leq n$, then*

$$\max_{|z|=1}|G_\gamma[P](z)| \geq \left\{\sum_{j=1}^s \gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^{n-s}\gamma_j}\left(\frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right)\right\}\max_{|z|=1}|P(z)|. \tag{2.3}$$

The result is sharp and equality holds for $P(z) = z^s(z+k)^{n-s}$, $s < n$. On setting $s = 0$ in Theorem 2.4, the following result is obtained.

Corollary 2.5. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \le k \le 1$,and $\gamma \in R^n_+$ with $\gamma_j \ge 1, 1 \le j \le n$, then*

$$\max_{|z|=1}|G_\gamma[P](z)| \ge \frac{\sum_{j=1}^n \gamma_j}{1+k}\left\{1 + \frac{k}{\sum_{j=1}^n \gamma_j}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{2.4}$$

The result is sharp and equality holds for $P(z) = (z+k)^n$. On setting $k = 1$, we obtain the following result, which is in fact the generalization of inequality (1.4) to the class of generalized derivative. In fact, we prove

Corollary 2.6. *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \le k \le 1$,and $\gamma \in R^n_+$ with $\gamma_j \ge 1, 1 \le j \le n$, then*

$$\max_{|z|=1}|G_\gamma[P](z)| \ge \frac{n}{2}\left\{1 + \frac{1}{n}\left(\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{2.5}$$

Remark 2.7. For a n-tuple $\gamma = (1, 1, \ldots, 1)$, Theorem 2.4, Corollary 2.5 and Corollary 2.6 reduce to Theorem 1.4 , Corollary 1.5 and inequality 1.4.

Next, we present the following result which generalizes Theorem 1.7 to the class of *generalized polar derivative.*

Theorem 2.8. *If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \le s \le n$, has all its zeros in $|z| \le k$, $k \le 1$, and $\gamma \in R^n_+$ with $\gamma_j \ge 1, 1 \le j \le n$, then for $\alpha \in C$ with $|\alpha| \ge k$,*

(2.6)

$$\begin{aligned}&\max_{|z|=1}\left|N^\gamma_\alpha[P](z)\right| \\ &\ge \big(|\alpha| - k\big)\left\{\sum_{j=1}^s \gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^{n-s}\gamma_j}\left(\frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right)\right\}\max_{|z|=1}|P(z)|.\end{aligned}$$

The result is sharp and equality holds for $P(z) = z^s(z+k)^{n-s}$, $s < n$, with $|\alpha| \ge k$. On setting $s = 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.9. *If $P(z) \in \mathcal{P}_n$, has all its zeros in $|z| \le k$, $k \le 1$, and $\gamma \in R^n_+$ with $\gamma_j \ge 1, 1 \le j \le n$, then for $\alpha \in C$ with $|\alpha| \ge k$,*

(2.7)
$$\max_{|z|=1}\left|N_\alpha^\gamma[P](z)\right| \geq \big(|\alpha|-k\big)\frac{\sum_{j=1}^{n}\gamma_j}{1+k}\left\{1+\frac{k}{\sum_{j=1}^{n}\gamma_j}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right\}\max_{|z|=1}|P(z)|.$$

The result is sharp and equality holds for $P(z)=(z+k)^n$, with $|\alpha|\geq k$.

Remark 2.10. For a n-tuple $\gamma=(1,1,\dots,1)$, Theorem 2.8 and Corollary 2.9 reduce to Theorem 1.7 and Corollary 1.8.

3. LEMMA

For the proof of Theorem 2.1 and Theorem 2.4 , we need the following lemmas.

Lemma 3.1. *If $0\leq x\leq 1$ and $\gamma\in R_+^n$ with $\gamma_j\geq 1, 1\leq j\leq n$, then*

$$\sum_{j=1}^{n}\frac{\gamma_j(1-x_j)}{1+x_j}\geq\frac{1-\prod_{j=1}^{n}x_j}{1+\prod_{j=1}^{n}x_j}, \qquad \forall\ n\in N. \tag{3.1}$$

Lemma 3.1 above is easily obtained by induction.

Lemma 3.2. *If $P(z)\in\mathcal{P}_n$, has all its zeros in $|z|\leq k$, $k\leq 1$, and $\gamma\in R_+^n$ with $\gamma_j\geq 1, 1\leq j\leq n$, then for $|z|=1$*

$$|G_\gamma[Q](z)|\leq k|G_\gamma[P](z)| \tag{3.2}$$

where $Q(z)=z^n\overline{P\big(1/\overline{z}\big)}$.

Proof. Since all the zeros of a polynomial P(z) lie in $|z|\leq k$, where $k\leq 1$, we can write

$$P(z)=a_n\prod_{j=1}^{n}(z-z_j), \qquad \text{where } |z_v|\leq k,\ j=1,2,\dots,n.$$

Since all the zeros of $P(z)$ are in $|z|\leq k$, then, $F(z)=P(kz)$ has all its zeros in$|z|\leq 1$, for $\gamma\in R_+^n$. This gives,

$$\frac{zG_\gamma[F](z)}{F(z)}=\sum_{j=1}^{n}\frac{\gamma_j z}{z-\zeta_j}, \qquad \text{where} \quad \zeta_j=\frac{z_j}{k} \quad \text{and} \quad |\zeta_j|\leq 1,\ 1\leq k\leq n. \tag{3.3}$$

Since for the points $e^{i\theta}$, $0\leq\theta\leq 2\pi$, other than the zeros of $F(z)$, we have

$$
\begin{aligned}
Re\Big\{\frac{e^{i\theta}G_\gamma[F](e^{i\theta})}{F(e^{i\theta})}\Big\} &= Re\Big\{\sum_{j=1}^{n}\frac{\gamma_j e^{i\theta}}{e^{i\theta}-\zeta_j}\Big\}\\
&= \sum_{j=1}^{n}\gamma_j Re\Big\{\frac{e^{i\theta}}{e^{i\theta}-\zeta_j}\Big\}\\
&\geq \frac{1}{2}\sum_{j=1}^{n}\gamma_j.
\end{aligned}
$$

Which implies,

$$
Re\left\{\frac{e^{i\theta}G_\gamma[F](e^{i\theta})}{(\sum_{j=1}^{n}\gamma_j)F(e^{i\theta})}\right\}\geq\frac{1}{2}.
$$

For the points $e^{i\theta},\ 0\leq\theta\leq 2\pi$, which are not the zeros of $F(z)$, we have

$$
\left|1-\frac{e^{i\theta}G_\gamma[F](e^{i\theta})}{(\sum_{j=1}^{n}\gamma_j)F(e^{i\theta})}\right|\leq\left|\frac{e^{i\theta}G_\gamma[F](e^{i\theta})}{(\sum_{j=1}^{n}\gamma_j)F(e^{i\theta})}\right|
$$

Equivalently

$$
\left|(\sum_{j=1}^{n}\gamma_j)F(e^{i\theta})-e^{i\theta}G_\gamma[F](e^{i\theta})\right|\leq\left|G_\gamma[F](e^{i\theta})\right| \tag{3.4}
$$

For the $e^{i\theta},\ 0\leq\theta\leq 2\pi$, which are not the zeros of $F(z)$. Since the inequality (3.4) is trivially true for points $e^{i\theta},\ 0\leq\theta\leq 2\pi$, which are zeros of $F(z)$, therefore, it follows that

$$
\left|\sum_{j=1}^{n}\gamma_j F(z)-zG_\gamma[F](z)\right|\leq\left|G_\gamma[F](z)\right| \quad\text{for}\quad |z|=1. \tag{3.5}
$$

Since $F(z)=P(kz)$, from (3.3) it follows that

$$
G_\gamma[F](z)=F(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-\frac{z_j}{k}}=kP(kz)\sum_{j=1}^{n}\frac{\gamma_j}{kz-z_j}=kG_\gamma[P](kz).
$$

Replacing $F(z)$ by $P(kz)$ and $G_\gamma[F](z)$ by $kG_\gamma[P](kz)$ in (3.5), we obtain

$$
\left|\sum_{j=1}^{n}\gamma_j P(kz)-zkG_\gamma[P](kz)\right|\leq k\left|G_\gamma[P](kz)\right| \quad\text{for}\quad |z|=1. \tag{3.6}
$$

Since $k\leq 1$, we take in particular $z=\frac{e^{i\theta}}{k},\ 0\leq\theta\leq 2\pi$ in (3.6) to get

$$
\left|\sum_{j=1}^{n}\gamma_j P(e^{i\theta})-e^{i\theta}G_\gamma[P](e^{\imath\theta})\right|\leq k\left|G_\gamma[P](e^{i\theta})\right|.
$$

This shows that,

$$\left|\sum_{j=1}^{n}\gamma_j P(z) - zG_\gamma[P](z)\right| \leq k\left|G_\gamma[P](z)\right| \quad \text{for} \quad |z| = 1. \tag{3.7}$$

Since

$$\begin{aligned}\sum_{j=1}^{n}\gamma_j P(z) - zG_\gamma[P](z) &= P(z)\sum_{j=1}^{n}\left(\gamma_j - \frac{z\gamma_j}{z-z_j}\right)\\ &= -P(z)\sum_{j=1}^{n}\left(\frac{\gamma_j z_j}{z-z_j}\right).\end{aligned} \tag{3.8}$$

Also

$$\begin{aligned}z^{n-1}\overline{G_\gamma[Q]\big(1/\overline{z}\big)} &= -z^n Q\big(1/\overline{z}\big)\sum_{j=1}^{n}\left(\frac{\gamma_j z_j}{z-z_j}\right)\\ &= -P(z)\sum_{j=1}^{n}\left(\frac{\gamma_j z_j}{z-z_j}\right).\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) , for $|z| = 1$ we have

$$\left|G_\gamma[Q](z)\right| = \left|\sum_{j=1}^{n}\gamma_j P(z) - zG_\gamma[P](z)\right|. \tag{3.10}$$

From (3.7) and (3.10) we have

$$\left|G_\gamma[Q](z)\right| \leq k\left|G_\gamma[P](z)\right| \quad \text{for} \quad |z| = 1. \tag{3.11}$$

□

4. PROOF OF MAIN RESULTS

Proof of Theorem 2.1. Since all the zeros of P(z) lie in $|z| \leq k$, where $k \leq 1$, we can write

$$P(z) = a_n \prod_{v=1}^{n}(z - z_j), \qquad \text{where} \quad |z_j| \leq k,\ j = 1, 2, \ldots, n.$$

Then, for $\gamma \in R_+^n$ and $P(z) \neq 0$ we have

$$\frac{G_\gamma[P](z)}{P(z)} = \sum_{j=1}^{n}\frac{\gamma_j}{z - z_j}.$$

Since for the points $e^{i\theta},\ 0 \leq \theta \leq 2\pi$, other than the zeros of $P(z)$, we have

$$
\begin{aligned}
Re\Big\{\frac{e^{i\theta}G_\gamma[P](e^{i\theta})}{P(e^{i\theta})}\Big\} &= Re\Big\{\sum_{j=1}^{n}\frac{\gamma_j e^{i\theta}}{e^{i\theta}-z_j}\Big\}\\
&= \sum_{j=1}^{n}\gamma_j Re\Big\{\frac{e^{i\theta}}{e^{i\theta}-z_j}\Big\}\\
&\geq \sum_{j=1}^{n}\frac{\gamma_j}{1-|z_j|}\\
&= \frac{\sum_{j=1}^{n}\gamma_j}{1+k}+\frac{k}{1+k}\sum_{j=1}^{n}\gamma_j\Big(\frac{k-|z_j|}{k+k|z_j|}\Big).
\end{aligned}
$$

Since $k \leq 1$, thus, $k + k|z_j| \leq k + |z_j|$ and $\gamma \in R_+^n$ with $\gamma_j \geq 1, \quad 1 \leq j \leq n$, for the points $e^{i\theta},\ 0 \leq \theta \leq 2\pi$, other than the zeros of $P(z)$, the above inequality yields

$$
Re\left\{\frac{e^{i\theta}G_\gamma[P](e^{i\theta})}{P(e^{i\theta})}\right\} \geq \frac{\sum_{j=1}^{n}\gamma_j}{1+k}+\frac{k}{1+k}\sum_{j=1}^{n}\gamma_j\left(\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right).
$$

By Lemma 3.1 and noting that $\dfrac{|z_j|}{k} \leq 1, \quad 1 \leq k \leq n$, we get for all z on $|z| = 1$, for which $P(z) \neq 0$

$$
\begin{aligned}
Re\left\{\frac{zG_\gamma[P](z)}{P(z)}\right\} &\geq \frac{\sum_{j=1}^{n}\gamma_j}{1+k}+\frac{k}{1+k}\left(\frac{1-\frac{\prod_{j=1}^{n}|z_j|}{k^n}}{1+\frac{\prod_{j=1}^{n}|z_j|}{k^n}}\right)\\
&= \frac{\sum_{j=1}^{n}\gamma_j}{1+k}\left(1+\frac{k}{\sum_{j=1}^{n}\gamma_j}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right).
\end{aligned}
$$

That proves Theorem 2.1. □

Proof of Theorem 2.4. Let $P(z) = z^s F(z) \in \mathcal{P}_n$, where $F(z) = a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}, \quad 0 \leq s \leq n$, then on $|z| = 1$,

$$
Re\left(\frac{zG_\gamma[P](z)}{P(z)}\right) = \sum_{j=1}^{s}\gamma_j + Re\left(\frac{zG_\gamma[F](z)}{F(z)}\right).
$$

Since $F(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leq k,\ k \leq 1$, therefore by Theorem 2.1 to F(z), for all points on$|z| = 1$, with other than the zeros of P(z), we have

$$
Re\left(\frac{zG_\gamma[P](z)}{P(z)}\right) \geq \sum_{j=1}^{s}\gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\left(1+\frac{k}{\sum_{j=1}^{n-s}\gamma_j}\left(\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\right).
$$

Which implies for all points z on $|z| = 1$, with other than the zeros of $P(z)$, we have

$$\left|\frac{zG_\gamma[P](z)}{P(z)}\right| \geq Re\left(\frac{zG_\gamma[P](z)}{P(z)}\right)$$
$$\geq \sum_{j=1}^{s}\gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^{n-s}\gamma_j}\left(\frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right).$$

which gives, for $|z| = 1$

(4.1)
$$\left|G_\gamma[P](z)\right| \geq \left\{\sum_{j=1}^{s}\gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^{n-s}\gamma_j}\left(\frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right)\right\}\left|P(z)\right|.$$

Which immediately leads to desired result. This proves Theorem 2.4. □

Proof of Theorem 2.8. Since $Q(z) = z^n\overline{P(1/\overline{z})}$, then from (3.10) we have

$$\left|G_\gamma[Q](z)\right| = \left|\sum_{j=1}^{n}\gamma_j P(z) - zG_\gamma[P](z)\right| \quad \text{for} \quad |z| = 1.$$

Since all the zeros of a polynomial P(z) lie in $|z| \leq k$, where $k \leq 1$, therefore by Lemma 3.2, we have

$$k\left|G_\gamma[P](z)\right| \geq \left|\sum_{j=1}^{n}\gamma_j P(z) - zG_\gamma[P](z)\right| \quad \text{for} \quad |z| = 1. \tag{4.2}$$

Now for every $\alpha \in C$ with $|\alpha| \geq k$, we have for $|z| = 1$,

$$\left|N_\alpha^\gamma[P](z)\right| = \left|\sum_{k=1}^{n}\gamma_k P(z) + (\alpha - z)G_\gamma[P](z)\right|$$
$$\geq \left|\alpha\right|\left|G_\gamma[P](z)\right| - \left|\sum_{j=1}^{n}\gamma_j P(z) - zG_\gamma[P](z)\right|$$

Using (4.2) we get

$$\left|N_\alpha^\gamma[P](z)\right| \geq \big(|\alpha| - k\big)\left|G_\gamma[P](z)\right| \quad \text{for} \quad |z| = 1 \tag{4.3}$$

Combining (4.1) and (4.3), we obtain for $|z| = 1$

$$\left|N_\alpha^\gamma[P](z)\right|$$
$$\geq \big(|\alpha| - k\big)\left\{\sum_{j=1}^{s}\gamma_j + \frac{\sum_{j=1}^{n-s}\gamma_j}{1+k}\left(1 + \frac{k}{\sum_{j=1}^{n-s}\gamma_j}\left(\frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right)\right\}\left|P(z)\right|.$$

Which immediately leads to desired result. This proves Theorem 2.8. □

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