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A REPRESENTATION FOR AN INVERSE GENERALIZED FOURIER-FEYNMAN TRANSFORM ASSOCIATED WITH GAUSSIAN PROCESS ON FUNCTION SPACE

Jae Gil Choi

Abstract. In this paper, we suggest a representation for an inverse transform of the generalized Fourier-Feynman transform on the function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$ is induced by the generalized Brownian motion process with mean function $a(t)$ and variance function $b(t)$. To do this, we study the generalized Fourier-Feynman transform associated with the Gaussian process $\mathcal{Z}_k$ of exponential-type functionals. We then establish that a composition of the $\mathcal{Z}_k$-generalized Fourier-Feynman transforms acts like an inverse generalized Fourier-Feynman transform.

1. Introduction

The present paper is an exposition of the elements of the Fourier–Feynman transform theory. Little originality can be claimed for the structures offered here, but our treatment is in several respects simpler and more direct than that in a number of sophisticated tools.

Let $C_0[0,T]$ denote one-parameter Wiener space. The study of the Fourier–Wiener transform of functionals on the infinite dimensional Banach space $C_0[0,T]$ was initiated by Cameron and Martin [2, 3, 4]. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on Euclidean space $\mathbb{R}^n$. Since then, many transforms which were somewhat analogous to the Fourier–Wiener transform have been defined and developed in the literature. There are two well-known transforms on the Wiener space $C_0[0,T]$. One of them is the 'analytic' Fourier–Feynman transform (FFT) [1, 5, 17] and the other is the integral transform (IT) [11, 18, 19, 20]. Each of the transforms on $C_0[0,T]$ has an inverse transform. For an elementary survey of these transforms, see [21].

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In [5, 17], the authors obtained the existence of the L_2 analytic FFT $T_q^{(2)}(F)$ for several large classes of functionals F on $C_0[0,T]$. In particular, they showed that for all real $q \neq 0$,

$$T_{-q}^{(2)}(T_q^{(2)}(F))(y) = F(y)$$

for scale-invariant almost every $y \in C_0[0,T]$. Thus L_2 analytic FFT "$T_q^{(2)}$" with parameter q has the inverse transform "$T_{-q}^{(2)}$". Also, in [11, 18], the authors studied the IT of functionals F in $L_2(C_0[0,T])$. The authors showed that for $F \in L_2(C_0[0,T])$ and nonzero complex numbers α and β with $|\beta| \leq 1$, $\beta \neq \pm 1$, $\mathrm{Re}(1-\beta^2) > 0$, $\alpha = \sqrt{1-\beta^2}$ and $-\pi/4 < \arg(\alpha) < \pi/4$,

$$\mathcal{F}_{\alpha',1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(y), \quad y \in C_0[0,T]$$

where $\alpha' = \sqrt{1-1/\beta^2}$. That is to say, "$\mathcal{F}_{\alpha,\beta}^{-1}$" is given by "$\mathcal{F}_{i\alpha/\beta,1/\beta}$".

In [8, 9, 10, 12, 14, 15], the authors generalized the two transforms, "FFT" and "IT", for functionals on the very general function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$, induced by a generalized Brownian motion process (GBMP), was introduced by Yeh [22, 23] and was used extensively in [6, 7, 8, 9, 10, 12, 13, 14, 15]. The Wiener process used in [1, 2, 3, 4, 5, 11, 17, 18, 19, 20] is stationary in time and is free of drift while the stochastic process used in this paper as well as in [6, 7, 8, 9, 10, 12, 13, 14, 15, 22], is nonstationary in time and is subject to a drift $a(t)$. However, when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the general function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$.

By an effect of drift "$a(t)$" of the GBMP, the generalized Feynman integral, the generalized FFT (GFFT) and the generalized IT (GIT) on $C_{a,b}[0,T]$ have unusual behaviors. For a more detailed study, see [6] and the references cited therein. Moreover, unfortunately, the GFFT on $C_{a,b}[0,T]$ has no inverse transform such as the FFT on $C_0[0,T]$. In order to discuss this problem, in [12], Chang, Chung and Skoug presented a version of inverse transform of the GIT $\mathcal{F}_{\alpha,\beta}$ as follows: for appropriate functionals F on $K_{a,b}[0,T]$, the complexification of $C_{a,b}[0,T]$,

$$\mathcal{F}_{-i\alpha,1}\mathcal{F}_{i\alpha,1}\mathcal{F}_{-\alpha/\beta,1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(y)$$

for $y \in K_{a,b}[0,T]$, i.e.,

$$\mathcal{F}_{\alpha,\beta}^{-1} = \mathcal{F}_{-i\alpha,1}\mathcal{F}_{i\alpha,1}\mathcal{F}_{-\alpha/\beta,1/\beta}.$$

On the other hand, the representation for an inverse transform of the 'analytic' GFFT have been studied [13, 15]. In order to express an inverse transform of the GFFT, the authors suggested two singular transforms $\mathcal{P}_q$ and $\mathcal{N}_q$ of functionals on

$C_{a,b}[0,T]$. They then established the facts that for almost every functional F on $C_{a,b}[0,T]$,

$$\mathcal{P}_q(F)(y) = T_q^{(2)}(F)(y) \text{ and } \mathcal{P}_q^{-1}(F)(y) = \mathcal{N}_{-q}(F)(y)$$

for scale-invariant almost every $y \in C_{a,b}[0,T]$. However, the inverse transforms of the GFFT investigated in [13, 15] are not analytic transforms.

Recently, in order to express the analytic inverse GFFT, Chang and the current author suggested other 'analytic' transform via the concept of the convolution product of functionals on $C_{a,b}[0,T]$ in [7]. In this paper, we suggest an inverse transform of the "GFFT associated with the Gaussian process $\mathcal{Z}_k$ ($\mathcal{Z}_k$-GFFT)" on the function space $C_{a,b}[0,T]$. The concept of the analytic transforms studied in this paper are not depend on the concept of the convolution product. But our general transforms studied in this paper involve the two analytic transforms studied in [7].

2. Preliminaries

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$ and $a'(t) \in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. The GBMP Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. For more details, see [6, 8, 9, 14, 22, 23]. By [23, Theorem 14.2], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-field of $C_{a,b}[0,T]$. We then complete this function space to obtain the measure space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all μ-Carathéodory measurable subsets of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\,\cdot\,)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho > 0$. If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., we write $F \approx G$. Note that the relation "$\approx$" is an equivalence relation.

Let $L_{a,b}^2[0,T]$ (see [9] and [14]) be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L_{a,b}^2[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < +\infty \text{ and } \int_0^T v^2(s)d|a|(s) < +\infty \right\}$$

where $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$. Then $L_{a,b}^2[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b} = \int_0^T u(t)v(t)dm_{|a|,b}(t) \equiv \int_0^T u(t)v(t)d[b(t) + |a|(t)],$$

where $m_{|a|,b}$ denotes the Lebesgue–Stieltjes measure induced by $|a|(\cdot)$ and $b(\cdot)$. In particular, note that $\|u\|_{a,b} \equiv \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore, $(L_{a,b}^2[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Next, let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L_{a,b}^2[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D : C'_{a,b}[0,T] \to L_{a,b}^2[0,T]$ be defined by the formula

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)db(t)$$

is also a separable Hilbert space.

Note that the two separable Hilbert spaces $L_{a,b}^2[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the linear operator given by equation (2.1).

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty. \tag{2.2}$$

Then, the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$. For more details, see [16]. Under the condition (2.2), we observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w,a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)db(t) = \int_0^T z(t)\frac{a'(t)}{b'(t)}db(t) = \int_0^T z(t)da(t).$$

3. Gaussian Processes

In order to present our results involving the analytic GFFT, we follow the exposition of [7, 8].

For each $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we let $(w,x)^{\sim}$ denote the Paley–Wiener–Zygmund (PWZ) stochastic integral. It is known that for each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w,x)^{\sim}$ exists for s-a.e. $x \in C_{a,b}[0,T]$. If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w,x)^{\sim}$ equals the Riemann–Stieltjes integral $\int_0^T z(t)dx(t)$. Furthermore, for each $w \in C'_{a,b}[0,T]$, $(w,x)^{\sim}$ is a Gaussian random variable with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. Also, we note that for $w, x \in C'_{a,b}[0,T]$, $(w,x)^{\sim} = (w,x)_{C'_{a,b}}$.

For each $t \in [0,T]$, let $\chi_{[0,t]}$ denote the indicator function of the interval $[0,t]$ and for $k \in C'_{a,b}[0,T]$ with $Dk = h$ and with $\|k\|_{C'_{a,b}} = [\int_0^T h^2(t)db(t)]^{1/2} > 0$, let $\mathcal{Z}_k(x,t)$ be the PWZ stochastic integral

$$\mathcal{Z}_k(x,t) = (D^{-1}(h\chi_{[0,t]}), x)^{\sim}. \tag{3.1}$$

Let

$$\gamma_k(t) = \int_0^t Dk(u)da(u) = \int_0^t h(u)da(u),$$

and let

$$\beta_k(t) = \int_0^t (Dk(u))^2 db(u) = \int_0^t h^2(u)db(u).$$

Then the stochastic process $\mathcal{Z}_k : C_{a,b}[0,T] \times [0,T] \to \mathbb{R}$ is Gaussian with mean function

$$\int_{C_{a,b}[0,T]} \mathcal{Z}_k(x,t)d\mu(x) = \int_0^t h(u)da(u) = \gamma_k(t)$$

and covariance function

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \big(\mathcal{Z}_k(x,s) - \gamma_k(s)\big)\big(\mathcal{Z}_k(x,t) - \gamma_k(t)\big)d\mu(x) \\
&= \int_0^{\min\{s,t\}} h^2(u)db(u) = \beta_k(\min\{s,t\}).
\end{aligned}$$

If $h = Dk$ is of bounded variation on $[0,T]$, then, for all $x \in C_{a,b}[0,T]$, $\mathcal{Z}_k(x,t)$ is continuous in t. Of course if $k(t) \equiv b(t)$, then $\mathcal{Z}_b(x,t) = x(t)$. Furthermore, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$ and the Gaussian process (3.1) with $k(t) \equiv t$ is an ordinary Wiener process.

Let $C^*_{a,b}[0,T]$ be the set of functions k in $C'_{a,b}[0,T]$ such that Dk is continuous except for a finite number of finite jump discontinuities and is of bounded variation on $[0,T]$. For any $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, let the operation $\odot$ between $C'_{a,b}[0,T]$ and $C^*_{a,b}[0,T]$ be defined by

$$w \odot k = D^{-1}(DwDk), \text{ i.e., } D(w \odot k) = DwDk,$$

where $DwDk$ denotes the pointwise multiplication of the functions Dw and Dk. In this case, $(C^*_{a,b}[0,T], \odot)$ forms a commutative algebra with the identity b. For more details, see [8].

Given any $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, it follows that

$$\begin{aligned}(w, \mathcal{Z}_k(x,\cdot))^\sim &= \int_0^T Dw(t) d\bigg(\int_0^t Dk(s)dx(s)\bigg)\\ &= \int_0^T Dw(t)Dk(t)dx(t)\\ &= (w \odot k, x)^\sim\end{aligned}$$

for s-a.e $x \in C_{a,b}[0,T]$. Thus, throughout the rest of this paper, we require k to be in $C^*_{a,b}[0,T]$ for each process $\mathcal{Z}_k$.

4. Generalized Fourier-Feynman Transform associated with Gaussian Paths

We define the $\mathcal{Z}_k$-function space integral (namely, the function space integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) for functionals F on $C_{a,b}[0,T]$ by the formula

$$I_k[F] \equiv I_{k,x}[F(\mathcal{Z}_k(x,\cdot))] = \int_{C_{a,b}[0,T]} F(\mathcal{Z}_k(x,\cdot))d\mu(x)$$

whenever the integral exists.

Throughout the rest of this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part, and nonzero complex numbers with nonnegative real part, respectively. Furthermore, for each $\lambda \in \widetilde{\mathbb{C}}$, $\lambda^{1/2}$ denotes the principal square root of λ; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda \in \widetilde{\mathbb{C}}_+$.

Let $\mathcal{Z}_k$ be the Gaussian process given by (3.1) and let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_{a,b}[0,T]$ such that the $\mathcal{Z}_k$-function space integral

$$J_F(\mathcal{Z}_k;\lambda) = I_{k,x}[F(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))] \tag{4.1}$$

exists and is finite for all $\lambda > 0$. If there exists a function $J_F^*(\mathcal{Z}_k;\lambda)$ analytic on $\mathbb{C}_+$ such that $J_F^*(\mathcal{Z}_k;\lambda) = J_F(\mathcal{Z}_k;\lambda)$ for all $\lambda \in (0,+\infty)$, then $J_F^*(\mathcal{Z}_k;\lambda)$ is defined to be the analytic $\mathcal{Z}_k$-function space integral (namely, the analytic function space integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$I_k^{\mathrm{an}_\lambda}[F] \equiv I_{k,x}^{\mathrm{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))] \equiv \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(\mathcal{Z}_k(x,\cdot))d\mu(x) = J_F^*(\mathcal{Z}_k;\lambda). \tag{4.2}$$

Next let F be a measurable functional whose analytic $\mathcal{Z}_k$-function space integral $I_k^{\mathrm{an}_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic $\mathcal{Z}_k$-Feynman integral (namely, the generalized analytic Feynman integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) of F with parameter q and we write

$$I_k^{\mathrm{anf}_q}[F] \equiv I_{k,x}^{\mathrm{anf}_q}[F(\mathcal{Z}_k(x,\cdot))] = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} I_{k,x}^{\mathrm{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))]. \tag{4.3}$$

We are now ready to state the definition of the analytic $\mathcal{Z}_k$-GFFT on function space.

Definition 4.1. Let $\mathcal{Z}_k$ be the Gaussian process given by (3.1) and let F be a scale-invariant measurable functional on $C_{a,b}[0,T]$ such that for all $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, the analytic $\mathcal{Z}_k$-function space transform

$$T_{\lambda,k}(F)(y) = I_{k,x}^{\mathrm{an}_\lambda}[F(y + \mathcal{Z}_k(x,\cdot))]$$

exists. For $p \in (1,2]$, we define the L_p analytic $\mathcal{Z}_k$-GFFT (namely, the GFFT associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$), $T_{q,k}^{(p)}(F)$ of F, by the formula,

$$T_{q,k}^{(p)}(F)(y) = \operatorname*{l.i.m.}_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,k}(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \int_{C_{a,b}[0,T]} \big|T_{\lambda,k}(F)(\rho y) - T_{q,k}^{(p)}(F)(\rho y)\big|^{p'} d\mu(y) = 0 \tag{4.4}$$

where $1/p + 1/p' = 1$. We define the L_1 analytic $\mathcal{Z}_k$-GFFT, $T_{q,k}^{(1)}(F)$ of F, by the formula

$$T_{q,k}^{(1)}(F)(y) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,k}(F)(y) = I_{k,x}^{\mathrm{anf}_q}[F(y + \mathcal{Z}_k(x,\cdot))] \tag{4.5}$$

if it exists.

We note that for $1 \le p \le 2$, $T_{q,k}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{q,k}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q,k}^{(p)}(G)$ exists and $T_{q,k}^{(p)}(G) \approx T_{q,k}^{(p)}(F)$. Moreover, from equations (4.3), (4.2) and (4.5), it follows that

$$I_k^{\mathrm{anf}_q}[F] \equiv I_{k,x}^{\mathrm{anf}_q}[F(\mathcal{Z}_k(x,\cdot))] = T_{q,k}^{(1)}(F)(0)$$

in the sense that if either side exists, then both sides exist and equality holds.

Remark 4.2. Note that if $k \equiv b$ on $[0,T]$, then the generalized analytic $\mathcal{Z}_b$-Feynman integral, $I_b^{\mathrm{anf}_q}[F]$, and the L_p analytic $\mathcal{Z}_b$-GFFT, $T_{q,b}^{(p)}(F)$ agree with the previous definitions of the generalized analytic Feynman integral and the analytic GFFT, respectively [7, 9, 14, 15].

5. Exponential-type Functionals

Let $\mathcal{E}$ be the class of all functionals Ψ_w which have the form

$$\Psi_w(x) = \exp\{(w,x)^{\sim}\} \tag{5.1}$$

for some $w \in C'_{a,b}[0,T]$ and for s-a.e. $x \in C_{a,b}[0,T]$. Given $q \in \mathbb{R}\setminus\{0\}$, $\tau \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, let $\mathcal{E}_{q,\tau,k}$ be the class of all functionals having the form

$$\Psi_w^{q,\tau,k}(x) = K_{q,\tau,k}^a \Psi_w(x) \tag{5.2}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where Ψ_w is given by equation (5.1) and $K_{q,\tau,k}^a$ is a complex number given by

$$K_{q,\tau,k}^a \equiv \exp\left\{\frac{i}{2q}\|\tau \odot k\|_{C'_{a,b}}^2 + (-iq)^{-1/2}(\tau \odot k, a)_{C'_{a,b}}\right\}. \tag{5.3}$$

The functionals given by equation (5.2) and linear combinations (with complex coefficients) of the $\Psi_w^{q,\tau,k}$'s are called the partially exponential-type functionals on $C_{a,b}[0,T]$. The functionals given by (5.1) are also partially exponential-type functionals because $\Psi_w^{q,\tau,0}(x) = \Psi_w^{q,0,k}(x) = \Psi_w(x)$ for s-a.e. $x \in C_0[0,T]$.

For notational convenience, let $\Psi_w^{0,\tau,k}(x) = \Psi_w(x)$ and let $\mathcal{E}_{0,\tau,k} = \mathcal{E}$. Then for any $(q,\tau,k) \in \mathbb{R} \times C'_{a,b}[0,T] \times C^*_{a,b}[0,T]$, the class $\mathcal{E}_{q,\tau,k}$ is dense in $L_2(C_{a,b}[0,T])$. Next, let $\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}$. Then, using the fact that

$$\mathcal{E} \equiv \mathcal{E}_{0,\tau,k} \subset \bigcup_{\substack{q\in\mathbb{R}\\ \tau\in C'_{a,b}[0,T]\\ k\in C^*_{a,b}[0,T]}} \mathcal{E}_{q,\tau,k} \subset \mathcal{E}(C_{a,b}[0,T]),$$

one can see that $\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}_{q,\tau,k}$ for every $(q,\tau,k) \in \mathbb{R} \times C'_{a,b}[0,T] \times C^*_{a,b}[0,T]$.

Note that every exponential-type functional is scale-invariant measurable. Since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0,T]$, $\mathcal{E}(C_{a,b}[0,T])$ can be regarded as the space of all s-equivalence classes of partially exponential-type functionals.

In our first theorem of this section, we give a formula for $\mathcal{Z}_k$-GFFT of exponential-type functionals.

Theorem 5.1. *Let $\Psi_w \in \mathcal{E}$ be given by equation* (5.1)*. Then for all $p \in [1,2]$, all real $q \neq 0$ and $k \in C^*_{a,b}[0,T]$ with $\|k\|_{C'_{a,b}} > 0$, the L_p analytic $\mathcal{Z}_k$-GFFT of Ψ_w, $T^{(p)}_{q,k}(\Psi_w)$ exists and is given by the formula*

$$T^{(p)}_{q,k}(\Psi_w) \approx \Psi^{q,w,k}_w, \tag{5.4}$$

where $\Psi^{q,w,k}_w$ is given by equation (5.2) *with τ replaced with w. Thus, $T^{(p)}_{q,k}(\Psi_w)$ is an element of $\mathcal{E}(C_{a,b}[0,T])$.*

It will be helpful to establish the following lemma before giving the proof of Theorem 5.1.

Lemma 5.2. *Let Ψ_w be given by* (5.1)*. Then for any $\rho > 0$, it follows that*

$$\int_{C_{a,b}[0,T]} \Psi_w(\rho\mathcal{Z}_k(x,\cdot))d\mu(x) = \exp\left\{\frac{\rho^2}{2}\|w \odot k\|^2_{C'_{a,b}} + \rho(w \odot k, a)_{C'_{a,b}}\right\}. \tag{5.5}$$

Proof of Theorem 5.1. Given $y \in C_{a,b}[0,T]$, first of all, using equations (4.1) with F replaced with $\Psi_w(y+\cdot)$, (5.1), and (5.5), it follows that for all $k \in C^*_{a,b}[0,T] \setminus \{0\}$ and every $\lambda > 0$,

$$\begin{aligned} J_{\Psi_w(y+\cdot)}(\mathcal{Z}_k;\lambda) &= I_{k,x}[\Psi_w(y+\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))] \\ &= \exp\{(w,y)^\sim\} \int_{C_{a,b}[0,T]} \Psi_w(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))d\mu(x) \\ &= \exp\left\{(w,y)^\sim + \frac{1}{2\lambda}\|w \odot k\|^2_{C'_{a,b}} + \lambda^{-1/2}(w \odot k, a)_{C'_{a,b}}\right\}. \end{aligned}$$

Thus, for all $\lambda > 0$ and s-a.e. $y \in C_{a,b}[0,T]$, we obtain the formula

$$T_{\lambda,k}(\Psi_w)(y) = \exp\left\{(w,y)^\sim + \frac{1}{2\lambda}\|w \odot k\|^2_{C'_{a,b}} + \lambda^{-1/2}(w \odot k, a)_{C'_{a,b}}\right\}.$$

But the last expression is analytic, as a function of λ, throughout $\mathbb{C}_+$ for s-a.e. $y \in C_{a,b}[0,T]$. Thus, in view of equation (4.5), it follows that $T^{(1)}_{q,k}(\Psi_w)$ exists and is given by the right hand side of (5.4) for all $q \in \mathbb{R} \setminus \{0\}$.

Next, for $\lambda \in \mathbb{C}_+$, $w \in C'_{a,b}[0,T]$ and $k \in C^*_{a,b}[0,T]$, let

$$L^a_{\lambda,w,k} \equiv \exp\left\{\frac{1}{2\lambda}\|w \odot k\|^2_{C'_{a,b}} + \lambda^{-1/2}(w \odot k, a)_{C'_{a,b}}\right\}.$$

Then, fixing $p \in (1,2]$, it follows that that for all $\rho > 0$ and all $\lambda \in \mathbb{C}_+$,

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \big|T_{\lambda,k}(\Psi_w)(\rho y) - \Psi^{q,w,k}_w(\rho y)\big|^{p'} d\mu(y) \\
&= \int_{C_{a,b}[0,T]} \exp\{\rho p'(w,y)^\sim\}\big|\big(L^a_{\lambda,w,k} - K^a_{q,w,k}\big)\big|^{p'} d\mu(y) \\
&= \big|\big(L^a_{\lambda,w,k} - K^a_{q,w,k}\big)\big|^{p'} \int_{C_{a,b}[0,T]} \exp\{\rho p'(w,y)^\sim\} d\mu(y) \\
&= \big|L^a_{\lambda,w,k} - K^a_{q,w,k}\big|^{p'} \exp\left\{\frac{\rho^2(p')^2}{2}\|w\|^2_{C'_{a,b}} + \rho p'(w,a)_{C'_{a,b}}\right\}.
\end{aligned}$$

Clearly, $L^a_{\lambda,w,k} \to K^a_{q,w,k}$, in the complete space $\mathbb{C}$, whenever $\lambda \to -iq$ through $\mathbb{C}_+$. Thus, in view of equation (4.4), we obtain equation (5.4) for all $p \in (1,2]$. $\square$

Theorem 5.3. *Given any $p \in [1,2]$, $q \in \mathbb{R} \setminus \{0\}$ and $k \in C^*_{a,b}[0,T]$ with $\|k\|_{C'_{a,b}} > 0$, the L_p analytic $\mathcal{Z}_k$-GFFT, $T^{(p)}_{q,k} : \mathcal{E}(C_{a,b}[0,T]) \to \mathcal{E}(C_{a,b}[0,T])$ is an onto transform.*

Proof. We first note that given any functional F in $\mathcal{E}(C_{a,b}[0,T])$, F can be written as

$$F \approx \sum_{j=1}^{n} c_j \Psi_{w_j} \tag{5.6}$$

for a finite sequence $\{w_1, \ldots, w_n\}$ in $C'_{a,b}[0,T]$ and a sequence $\{c_1, \ldots, c_n\}$ in $\mathbb{C} \setminus \{0\}$ since $\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}$. We next note that for every $(q,w,k) \in \mathbb{R} \times C'_{a,b}[0,T] \times C^*_{a,b}[0,T]$, the complex number $K^a_{q,w,k}$ given by (5.3) with τ replaced with w is nonzero. Thus, using the linearity of the analytic $\mathcal{Z}_k$-GFFT $T^{(p)}_{q,k}$, (5.4), (5.3), and (5.2), it follows that for every $(q,w,k) \in \mathbb{R} \times C'_{a,b}[0,T] \times C^*_{a,b}[0,T]$,

$$T^{(p)}_{q,k}((K^a_{q,w,k})^{-1}\Psi_w) \approx (K^a_{q,w,k})^{-1} T^{(p)}_{q,k}(\Psi_w) \approx (K^a_{q,w,k})^{-1}\Psi^{q,w,k}_w(x) \approx \Psi_w$$

where $(K^a_{q,w,k})^{-1}$ denotes the reciprocal number of $K^a_{q,w,k}$. Using this and the linearity of $T^{(p)}_{q,k}$, again, it follows that for every functional $F \in \mathcal{E}(C_{a,b}[0,T])$ given by equation (5.6),

$$T_{q,k}^{(p)}\bigg(\sum_{j=1}^{n} c_j (K_{q,w_j,k}^{a})^{-1}\Psi_{w_j}\bigg) \approx F.$$

Hence the theorem is proved. □

6. A Representation for the Inverse GFFT

In this section, we present a representation of the inverse transform of the $\mathcal{Z}_k$-GFFT for functionals F in $\mathcal{E}(C_{a,b}[0,T])$. To do this we first investigate the iterated GFFT associated with Gaussian paths.

Lemma 6.1 below follows easily from (5.4), (5.2), and the fact that the space $\mathcal{E}(C_{a,b}[0,T])$ is the linear span of the exponential-type functionals.

Lemma 6.1. *Let $(k_1,\ldots,k_m)$ be a sequence of nonzero functions in $C_{a,b}^*[0,T]$, and let $\Psi_w \in \mathcal{E}$ be given by (5.1). Then for all $p\in[1,2]$ and all real numbers $q_1,\ldots,q_m$, the iterated analytic GFFT of Ψ exists and is given by the formula*

$$T_{q_m,k_m}^{(p)}\big(T_{q_{m-1},k_{m-1}}^{(p)}\big(\cdots\big(T_{q_1,k_1}^{(p)}(\Psi_w)\big)\cdots\big)\big)(y) = \exp\{(w,y)^{\sim}\}\bigg(\prod_{l=1}^{m} K_{q_l,w,k_l}^{a}\bigg)$$

for s-a.e. $y\in C_{a,b}[0,T]$, where K_{q_l,w,k_l}^{a} is given by equation (5.3) with (q,τ,k) replaced with (q_l,w,k_l) for each $l\in\{1,\ldots,m\}$.

The observations in (1) and (2) below will be very useful in the representation of GFFTs associated Gaussian processes.

(1) Let F be an element of $\mathcal{E}(C_{a,b}[0,T])$. Applying equation (5.6), one can see that the iterated analytic GFFT of F, $T_{q_m,k_m}^{(p)}(T_{q_{m-1},k_{m-1}}^{(p)}(\cdots(T_{q_1,k_1}^{(p)}(F))\cdots))$, exists. Thus for each functional $F\in\mathcal{E}(C_{a,b}[0,T])$ given by equation (5.6), it follows that

$$\begin{aligned}&T_{q_m,k_m}^{(p)}\big(T_{q_{m-1},k_{m-1}}^{(p)}\big(\cdots\big(T_{q_1,k_1}^{(p)}(F)\big)\cdots\big)\big)(y)\\&=\exp\{(w,y)^{\sim}\}\sum_{j=1}^{n} c_j\bigg(\prod_{l=1}^{m} K_{q_l,w_j,k_l}^{a}\bigg)\end{aligned}\tag{6.1}$$

for s-a.e. $y\in C_{a,b}[0,T]$.

(2) Let $p\in[1,2]$ be fixed and let $\Psi_w\in\mathcal{E}$ be given by (5.1). Then, in view of Theorem 5.1, we can see that for all nonzero real q, the analytic GFFTs,

$$T_{q,k}^{(p)}(\Psi_w),\quad T_{-q,k}^{(p)}(T_{q,k}^{(p)}(\Psi_w)),\quad T_{q,-k}^{(p)}(T_{-q,k}^{(p)}(T_{q,k}^{(p)}(\Psi_w))),$$

and

$$T^{(p)}_{-q,-k}(T^{(p)}_{q,-k}(T^{(p)}_{-q,k}(T^{(p)}_{q,k}(\Psi_w))))$$

all exist. Furthermore, it follows that

$$\begin{aligned}&T^{(p)}_{-q,-k}\big(T^{(p)}_{q,-k}\big(T^{(p)}_{-q,k}\big(T^{(p)}_{q,k}(\Psi_w)\big)\big)\big)(y)\\&= \exp\{(w,y)^\sim\}K^a_{-q,w,-k}K^a_{q,w,-k}K^a_{-q,w,k}K^a_{q,w,k}\\&= \Psi_w(y)\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

In view of Lemma 6.1 and the observation above, we have the following assertion.

Theorem 6.2. *Let $p \in [1,2]$ be given and let F be an element of $\mathcal{E}(C_{a,b}[0,T])$. Then for all $k \in C^*_{a,b}[0,T] \setminus \{0\}$ and all nonzero real q,*

$$T^{(p)}_{-q,-k}\big(T^{(p)}_{q,-k}\big(T^{(p)}_{-q,k}(T^{(p)}_{q,k}(F))\big)\big) \approx F. \tag{6.2}$$

Thus we have

$$\{T^{(p)}_{q,k}\}^{-1} = T^{(p)}_{-q,-k} \circ T^{(p)}_{q,-k} \circ T^{(p)}_{-q,k}.$$

Moreover, we have the six possibilities for the inverse transform of $T^{(p)}_{q,k}$:

$$\begin{aligned}&T^{(p)}_{-q,-k} \circ T^{(p)}_{q,-k} \circ T^{(p)}_{-q,k} = T^{(p)}_{-q,-k} \circ T^{(p)}_{-q,k} \circ T^{(p)}_{q,-k}\\&= T^{(p)}_{q,-k} \circ T^{(p)}_{-q,-k} \circ T^{(p)}_{-q,k} = T^{(p)}_{q,-k} \circ T^{(p)}_{-q,k} \circ T^{(p)}_{-q,-k}\\&= T^{(p)}_{-q,k} \circ T^{(p)}_{-q,-k} \circ T^{(p)}_{q,-k} = T^{(p)}_{-q,k} \circ T^{(p)}_{q,-k} \circ T^{(p)}_{-q,-k}\end{aligned} \tag{6.3}$$

on $\mathcal{E}(C_{a,b}[0,T])$.

7. Comments with the Previous Works

7.1. With the previous work on the function space $C_{a,b}[0,T]$: As mentioned in Remark 4.2, the L_p analytic GFFT $T^{(p)}_q$ studied in [9, 14, 15] can be considered as the L_p analytic $\mathcal{Z}_b$-GFFT $T^{(p)}_{q,b}$. In [7], to obtain an inverse transform of the L_1 analytic $\mathcal{Z}_b$-GFFT, the authors defined an L_1-type transform T^-_q. Given a functional F on $C_{a,b}[0,T]$, the L_1 analytic transform $T^-_q(F)$ was defined as follows:

$$T^-_q(F)(y) = \int^{\mathrm{anf}_q}_{C_{a,b}[0,T]} F(y-x)d\mu(x).$$

This deliberate structure is suggested on the fact that

$$\int_{C_{a,b}[0,T]} F(x)d\mu(x) \neq \int_{C_{a,b}[0,T]} F(-x)d\mu(x)$$

for almost every functional F on $C_{a,b}[0,T]$. However, in view of Definition 4.1, one can see that

$$T_q^{(1)}(F) = T_{q,b}^{(1)}(F) \tag{7.1}$$

and

$$T_q^{-}(F) = T_{q,-b}^{(1)}(F) \tag{7.2}$$

for functionals F on $C_{a,b}[0,T]$. Furthermore, when we use the right hand sides of equations (7.1) and (7.2), respectively, we can construct an inverse transform of the L_p analytic GFFT, $T_q^{(p)}(F)$ of F in $\mathcal{E}(C_{a,b}[0,T])$, as presented Theorem 6.2, for all $p \in [1,2]$.

7.2. With the previous work on the Wiener space $C_0[0,T]$: In the case that $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$. In this case, it also follows that

$$\begin{aligned} C'_{a,b}[0,T] &\equiv C'_0[0,T] \\ &= \left\{\tau : \tau(t) = \int_0^t v(s)ds \text{ for some } v \in L^2[0,T]\right\} \\ &= \{\tau : \tau \text{ is absolutely continuous on } [0,T] \text{ with } D\tau(0) = 0\} \end{aligned}$$

and

$$\begin{aligned} C^*_{a,b}[0,T] \equiv C^*_0[0,T] = \{k \in C^*_0[0,T] : & Dk \text{ is right continuous and} \\ & \text{of bounded variation on } [0,T]\} \end{aligned}$$

where D is the operator given by equation (2.1) with $b(t) = t$. Furthermore, one can see that the complex number $K^a_{q,\tau,k}$ given by (5.3) can be rewritten by

$$K^0_{q,\tau,k} \equiv \exp\left\{\frac{i}{2q}\|\tau \odot k\|^2_{C'_0}\right\}$$

for all nonzero real q, $\tau \in C'_0[0,T]$ and $k \in C^*_0[0,T]$.

Using this, one can see that for all nonzero real q, all $p \in [1,2]$, all $k \in C^*_0[0,T]$ with $\|k\|_{C'_0} > 0$, and every $\Psi_w \in \mathcal{E}$ (and hence every $F \in \mathcal{E}(C_0[0,T])$),

$$\begin{aligned} &T^{(p)}_{-q,-k}\big(T^{(p)}_{q,-k}\big(T^{(p)}_{-q,k}(\Psi_w)\big)\big)(y) \\ &= \exp\{(w,y)^{\sim}\}K^0_{-q,w,-k}K^0_{q,w,-k}K^0_{-q,w,k} \\ &= \exp\{(w,y)^{\sim}\}K^0_{-q,w,k} \\ &= T^{(p)}_{-q,k}(\Psi_w)(y) \end{aligned}$$

for s-a.e. $y \in C_0[0,T]$. Thus, in view of equation (6.2), it follows that

$$T^{(p)}_{-q,k}(T^{(p)}_{q,k}(F)) \approx T^{(p)}_{-q,-k}\big(T^{(p)}_{q,-k}\big(T^{(p)}_{-q,k}(T^{(p)}_{q,k}(F))\big)\big) \approx F, \tag{7.3}$$

i.e.,

$$\{T^{(p)}_{q,k}\}^{-1}(F) \approx T^{(p)}_{-q,k}(F)$$

for every exponential-type functionals F in $\mathcal{E}(C_0[0,T])$.

On the other hand, given an exponential-type functional F in $\mathcal{E}(C_0[0,T])$, one can also see that for all nonzero real q and all nonzero functions k in $C_0^*[0,T]$,

$$T^{(p)}_{q,k}(F) \approx T^{(p)}_{q,-k}(F).$$

Using this and applying (6.1) with $K^a_{q_l,w_j,k_l}$ replaced with $K^0_{q_l,w_j,k_l}$, it follows that

$$T^{(p)}_{q,-k}(T^{(p)}_{q,k}(F)) \approx T^{(p)}_{q/2,k}(F) \tag{7.4}$$

for all $F \in \mathcal{E}(C_{a,b}[0,T])$. Next using the second expression of (6.3), (7.4) and (7.3), it follows that

$$\begin{aligned}
&T^{(p)}_{-q,-k}\big(T^{(p)}_{q,-k}\big(T^{(p)}_{-q,k}\big(T^{(p)}_{q,k}(F)\big)\big)\big)\\
&\approx T^{(p)}_{-q,-k}\big(T^{(p)}_{-q,k}\big(T^{(p)}_{q,-k}\big(T^{(p)}_{q,k}(F)\big)\big)\big)\\
&\approx T^{(p)}_{-q/2,-k}\big(T^{(p)}_{q/2,k}(F)\big)\\
&\approx F
\end{aligned}$$

for all functionals F in $\mathcal{E}(C_0[0,T])$.

Frankly speaking, the fundamental relations (7.3) and (7.4) are based on the fact that

$$\int_{C_0[0,T]} F(x)dm_w(x) = \int_{C_0[0,T]} F(-x)dm_w(x)$$

for every measurable functional F on $C_0[0,T]$.

Remark 7.1. In view of the simple survey above, we can emphasize that the drift term $a(t)$ of the GBMP plays a prominent role in the existence of the inverse GFFT for the functionals F of the paths of the GBMP.

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SCHOOL OF GENERAL EDUCATION, DANKOOK UNIVERSITY, CHEONAN 31116, REPUBLIC OF KOREA
Email address: jgchoi@dankook.ac.kr