

THE BRUHAT ORDER OF GENERALIZED ALTERNATING SIGN MATRICES AND ITS RANK

HWA KYUNG KIM

ABSTRACT. We continue the investigations in [7] extending the Bruhat order on $n \times n$ alternating sign matrices to a more general setting. We show that the resulting partially ordered set is a lattice and also investigate its rank.

1. INTRODUCTION

An $n \times n$ *alternating sign matrix* (abbreviated to ASM) is a $(0, \pm 1)$ matrix such that, ignoring 0's, the +1's and in each row and each column alternate beginning and ending with +1. The origins and properties of ASMs can be found in [1].

In [7] a generalization of alternating sign matrices was defined as follows: Let $u = (u_1, u_2, \dots, u_n)$, $u' = (u'_1, u'_2, \dots, u'_n)$, $v = (v_1, v_2, \dots, v_m)$, and $v' = (v'_1, v'_2, \dots, v'_m)$ be vectors of ± 1 's. If A is an $m \times n$ matrix, we define $A(u, u'|v, v')$ to be the $(m+2) \times (n+2)$ matrix (1.1) with rows indexed by $0, 1, \dots, m+1$ and columns indexed by $0, 1, \dots, n+1$.

$$(1.1) \quad A(u, u'|v, v') = \begin{array}{c|cccccc|c} 0 & u_1 & u_2 & \cdots & u_{n-1} & u_n & 0 \\ \hline v_1 & & & & & & v'_1 \\ v_2 & & & & & & v'_2 \\ \vdots & & & A & & & \vdots \\ v_{m-1} & & & & & & v'_{m-1} \\ v_m & & & & & & v'_m \\ \hline 0 & u'_1 & u'_2 & \cdots & u'_{n-1} & u'_n & 0 \end{array}.$$

We then write $A = A(u, u'|v, v')[1, 2, \dots, m|1, 2, \dots, n]$ to denote that A is the middle $m \times n$ submatrix of $A(u, u'|v, v')$, and also denote this by $A(u, u'|v, v') \rightarrow A$.

A $(u, u'|v, v')$ -ASM is an $m \times n$ $(0, \pm 1)$ -matrix A as in (1.1) such that, ignoring 0's, the +1's and -1 's in rows $0, 1, 2, \dots, m+1$ and columns $0, 1, 2, \dots, n+1$ of

Received by the editors September 27, 2023. Revised February 14, 2024. Accepted Feb. 18, 2024.
 2020 *Mathematics Subject Classification.* 05B20, 15B48.

Key words and phrases. generalized alternating sign matrix, Bruhat order.

the $(0, \pm 1)$ -matrix $A(u, u'|v, v')$ alternate. Thus the condition that the first and last nonzero entry in rows and columns of an ASM is $+1$ is relaxed, where now the first and last nonzero of each row is determined by v and v' , respectively, and the first and last nonzero of each column is determined by u and u' , respectively. Note also that, unlike ASMs, the first and last rows and columns of a $(u, u'|v, v')$ -ASM depend on u, u', v, v' and so may contain more than one nonzero entry. We denote the set of $(u, u'|v, v')$ -ASMs by $\mathcal{A}_{m,n}(u, u'|v, v')$. If $u = u'$ and $v = v'$ and $m = n$, we often abbreviate these notations to: (u, v) -ASM and $\mathcal{A}_n(u, v)$, respectively. If $u = u' = v = v' = (-1, -1, \dots, -1)$, we also use the abbreviations (u) -ASM and $\mathcal{A}_n(u)$. Observe that if $u = u' = v = v' = (-1, -1, \dots, -1)$, then a (u) -ASM is an ordinary ASM, and $\mathcal{A}_n(u)$ is the usual set $\mathcal{A}_n$ of $n \times n$ ASMs.

Example 1.1. Let $m = 2$ and $n = 3$, and let $u = (1, -1, 1)$, $u' = (-1, 1, -1)$, $v = (1, -1)$ and $v = (1, -1)$. Then a $(u, u'|v, v')$ -ASM is given below:

$$\left[\begin{array}{c|ccc|c} 0 & 1 & -1 & 1 & 0 \\ \hline 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ \hline 0 & -1 & 1 & -1 & 0 \end{array} \right] \rightarrow \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

In [7], necessary and sufficient conditions are given for the nonemptiness of $\mathcal{A}_{m,n}(u, u'|v, v')$. Let $u = (u_1, u_2, \dots, u_n)$, $u' = (u'_1, u'_2, \dots, u'_n)$, $v = (v_1, v_2, \dots, v_m)$, and $v' = (v'_1, v'_2, \dots, v'_m)$ be vectors of ± 1 s. We define m -vectors $r^+(v, v') = (r_1^+, r_2^+, \dots, r_m^+)$ and $r^-(v, v') = (r_1^-, r_2^-, \dots, r_m^-)$, and n -vectors $c^+(u, u') = (c_1^+, c_2^+, \dots, c_n^+)$ and $c^-(u, u') = (c_1^-, c_2^-, \dots, c_n^-)$ as follows:

- $r_k^+ = r_k^+(v, v')$ is the number of $i \leq k$ such that $v_i = v'_i = +1$;
- $r_k^- = r_k^-(v, v')$ is the number of $i \leq k$ such that $v_i = v'_i = -1$;
- $c_l^+ = c_l^+(u, u')$ is the number of $j \leq l$ such that $u_j = u'_j = +1$; and
- $c_l^- = c_l^-(u, u')$ is the number of $j \leq l$ such that $u_j = u'_j = -1$.

In particular, we have:

- $r_m^+ = r_m^+(v, v')$ is the total number of i such that $v_i = v'_i = +1$;
- $r_m^- = r_m^-(v, v')$ is the total number of i such that $v_i = v'_i = -1$;
- $c_n^+ = c_n^+(u, u')$ is the total number of j such that $u_j = u'_j = +1$; and
- $c_n^- = c_n^-(u, u')$ is the total number of j such that $u_j = u'_j = -1$.

$$(1.2) \quad r_m^-(v, v') - r_m^+(v, v') = c_n^-(u, u') - c_n^+(u, u').$$

We set

$$\begin{aligned} u^+ &= |\{j : u_j = +1\}| \text{ and } u^- = |\{j : u_j = -1\}|, \\ v^+ &= |\{i : v_i = +1\}| \text{ and } v^- = |\{i : v_i = -1\}|. \end{aligned}$$

If we consider the sum of the first k rows of $(u, u'|v, v')$ -ASM, then we obtain

$$(1.3) \quad -u^+ \leq r_k^-(v, v') - r_k^+(v, v') \leq u^- \quad (k = 1, 2, \dots, m).$$

In a similar way, we obtain

$$(1.4) \quad -v^+ \leq c_l^-(u, u') - c_l^+(u, u') \leq v^- \quad (l = 1, 2, \dots, n).$$

In [7], a $(u, u'|v, v')$ -ASM exists if and only if (1.2), (1.3), and (1.4) hold. In this paper we primarily consider $\mathcal{A}(u, u'|v, v')$ where (1.2), (1.3), and (1.4) hold.

For any $m \times n$ matrix $A = [a_{ij}]$, the *sum-matrix* $\Sigma(A) = [\sigma_{ij}]$ of A is the $m \times n$ matrix where

$$\sigma_{ij} = \sigma_{ij}(A) = \sum_{k \leq i} \sum_{l \leq j} a_{kl} \quad (1 \leq i \leq m, 1 \leq j \leq n),$$

the sum of the entries in the leading $i \times j$ submatrix of A . Define $\sigma_{ij} = 0$ if $i = 0$ or $j = 0$. Then the matrix A is uniquely determined by its sum-matrix $\Sigma(A)$, namely,

$$a_{ij} = \sigma_{ij} - \sigma_{i,j-1} - \sigma_{i-1,j} + \sigma_{i-1,j-1} \quad (1 \leq i \leq m, 1 \leq j \leq n).$$

The sum-matrix $\Sigma(A) = [\sigma_{ij}]$ of a matrix $A \in \mathcal{A}(u, u'|v, v')$ has the following properties which are easily verified:

- The integers in row i and column j of $\Sigma(A)$ are taken from the set $\{0, \pm 1, \pm 2, \dots\}$; moreover, $\sigma_{i1} \in \{0, -v_i\}$ and $\sigma_{1j} \in \{0, -u_j\}$ so that $\sigma_{i1} \in \{0, \pm 1\}$ for $1 \leq i \leq m$ and $\sigma_{1j} \in \{0, \pm 1\}$ for $1 \leq j \leq n$.
- Consecutive entries in a row or column differ in absolute value by at most 1.
- $\sigma_{in} = -(v_1 + v_2 + \dots + v_i)$ for $1 \leq i \leq m$.
- $\sigma_{nj} = -(u_1 + u_2 + \dots + u_j)$ for $1 \leq j \leq n$.

Example 1.2. Let $u = u' = (1, -1, 1, -1, 1)$ and $v = v' = (1, -1, 1)$. For

$$A = \left[\begin{array}{c|c|c|c|c} & & & & -1 \\ \hline & & & & \\ \hline & 1 & -1 & 1 & \\ \hline -1 & & & & \end{array} \right] \in \mathcal{A}_{3,5}(u, v), \text{ we have } \Sigma(A) = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ \hline -1 & 0 & -1 & 0 & -1 \end{array} \right].$$

In the next section we generalize the Bruhat order on the set $\mathcal{A}_n$ of $n \times n$ ASMs to the set $\mathcal{A}_{m,n}(u, u'|v, v')$. As with $\mathcal{A}_n$ we obtain a ranked lattice. In [8], we have $\mathcal{A}_n(u, v)$ is a ranked lattice where $\mathcal{A}_n(u, v)$ denote $\mathcal{A}_{m,n}(u, u'|v, v')$ for $u = u' =$

$(u_1, u_2, \dots, u_n)$ and $v = v' = (v_1, v_2, \dots, v_n)$ which are vectors of ± 1 's where u and v have the same number of $+1$'s and the same number of -1 's.

2. BRUHAT ORDER AND LATTICE

There is a partial order, called the *Bruhat order* and denoted by $\preceq_B$, defined on the set $\mathcal{S}_n$ of permutations of $\{1, 2, \dots, n\}$, equivalently, the set $\mathcal{P}_n$ of $n \times n$ permutation matrices, which has also been extended to the set $\mathcal{A}_n$ of $n \times n$ ASMs [10]. We briefly describe this partial order in its various equivalent forms [8].

- (i) For $\pi, \tau \in \mathcal{S}_n$, $\pi \preceq_B \tau$ provided π can be obtained from τ by a sequence of transpositions each of which reduces the number of inversions, not necessarily the set of inversions. There is such a sequence of transpositions each of which reduces the number of inversions by exactly one (but not, in general, by removing one inversion). The identity $\iota_n = (1, 2, \dots, n)$ (the identity matrix I_n using $\mathcal{P}_n$) is the unique minimal permutation in the Bruhat order on $\mathcal{S}_n$; the anti-identity $\zeta_n = (n, \dots, 2, 1)$ (the anti-identity matrix L_n using $\mathcal{P}_n$) is the unique maximal permutation. In terms of permutation matrices, for $P, Q \in \mathcal{P}_n$, $P \preceq_B Q$ if and only if P can be obtained from Q by a sequence of *interchanges* involving 2×2 submatrices (not necessarily with consecutive rows and consecutive columns):

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- (ii) Another characterization of the Bruhat order is: For $P, Q \in \mathcal{P}_n$, $P \preceq_B Q$ if and only if $\Sigma(P) \geq \Sigma(Q)$ (entrywise).
- (iii) The Bruhat order extends to $\mathcal{A}_n$ by defining for $A_1, A_2 \in \mathcal{A}_n$, $A_1 \preceq_B A_2$ provided that $\Sigma(A_1) \geq \Sigma(A_2)$. Then $(\mathcal{A}_n, \preceq_B)$ is a graded lattice extending the partially ordered set $(\mathcal{P}_n, \preceq_B)$, and indeed is the (unique up to isomorphism) smallest lattice extending $(\mathcal{P}_n, \preceq_B)$ (the *Dedekind-MacNeille completion* of $(\mathcal{P}_n, \preceq_B)$) [10]. The minimal element of $(\mathcal{A}_n, \preceq_B)$ is I_n , and the maximal element is L_n .
- (iv) For $A_1, A_2 \in \mathcal{A}_n$, $A_1 \preceq_B A_2$ if and only if there is a sequence of ASMs, $X_1 = A_1, X_2, \dots, X_p = A_2$, such that for $t = 1, 2, \dots, p-1$, X_t can be obtained from X_{t+1} by an *interchange* which adds T where the 2×2 submatrix

$T[i, j|k, l]$ ($i < j, k < l$) of T is the 2×2 matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and all other entries are zero. In order that the resulting matrices be ASMs, the 2×2 submatrix of X_{t+1} must equal

$$\left[\begin{array}{c|c} 0 \text{ or } -1 & 0 \text{ or } 1 \\ \hline 0 \text{ or } 1 & 0 \text{ or } -1 \end{array} \right].$$

An interchange is the analogue for ASMs of an interchange of a permutation matrix.

- (v) Let $P, Q \in \mathcal{P}_n$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{P}_n$ is defined by $P \preceq_b Q$ provided P can be obtained from Q by a sequence of *adjacent interchanges*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

applied to 2×2 submatrices with consecutive rows and consecutive columns. Since interchanges reduce the number of inversions of the corresponding permutations by exactly 1, $(\mathcal{P}_n, \preceq_b)$ is a subpartially ordered set of $(\mathcal{P}_n, \preceq_B)$ and, in fact, $(\mathcal{P}_n, \preceq_b)$ is a lattice. This lattice can be equivalently described in terms of the corresponding permutations in $\mathcal{S}_n$ as follows: For $\pi_1, \pi_2 \in \mathcal{S}_n$, $\pi_1 \preceq_b \pi_2$ if and only if the set $\text{inv}(\pi_1)$ of inversions of π_1 is a subset of the set $\text{inv}(\pi_2)$ of inversions of π_2 . In general, $\text{inv}(\pi_1) \cap \text{inv}(\pi_2)$ is the set of inversions of a unique permutation $\pi_1 \wedge \pi_2$, the *meet* of π_1 and π_2 , and $\text{inv}(\pi_1) \cup \text{inv}(\pi_2)$ is the set of inversions of a unique permutation $\pi_1 \vee \pi_2$, the *join* of π_1 and π_2 [2].

- (vi) Let $A_1, A_2 \in \mathcal{A}_n$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{A}_n$ can be defined by writing $A_1 \preceq_b A_2$ provided A_1 can be obtained from A_2 by a sequence of *adjacent interchanges* each of which adds to a 2×2 submatrix $A_2[i, i+1|j, j+1]$ of A_2 with consecutive rows and columns, the $n \times n$ matrix $T_{i,j}$ which is all 0's except for its 2×2 submatrix determined by rows i and $i+1$ and columns j and $j+1$ which equals

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and where the result is also an ASM. Unlike for $\mathcal{P}_n$, the Bruhat order and weak Bruhat order coincide on $\mathcal{A}_n$.

- (vii) Let $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ be vectors of ± 1 's where u and v have the same number of $+1$'s and the same number of -1 's. Let

$A_1, A_2 \in \mathcal{A}_n(u, v)$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{A}_n(u, v)$ can be defined by writing $A_1 \preceq_b A_2$ provided A_1 can be obtained from A_2 by a sequence of *adjacent interchanges* each of which adds to a 2×2 submatrix $A_2[i, i+1|j, j+1]$ of A_2 with consecutive rows and columns, the $n \times n$ matrix $T_{i,j}$ which is all 0's except for its 2×2 submatrix determined by rows i and $i+1$ and columns j and $j+1$ which equals

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and where the result is also an element in $\mathcal{A}_n(u, v)$.

These and other properties of ASMs can be found in several places including [2, 3, 4, 5, 6, 7, 8, 9, 10].

Let $u = (u_1, u_2, \dots, u_n)$, $u' = (u'_1, u'_2, \dots, u'_n)$, $v = (v_1, v_2, \dots, v_m)$, and $v' = (v'_1, v'_2, \dots, v'_m)$ be vectors of ± 1 's. There is a partial order, called the *Bruhat order* and denoted by $\preceq_B$, on $\mathcal{A}_{m,n}(u, v|u', v')$ of $(u, u'|v, v')$ -ASMs. There are three possibilities for defining $A_1 \preceq_B A_2$ for $A_1, A_2 \in \mathcal{A}_{m,n}(u, v|u', v')$, namely:

- (a) $\Sigma(A_1) \geq \Sigma(A_2)$ (entrywise).
- (b) A_1 can be obtained from A_2 by a sequence of interchanges.
- (c) A_1 can be obtained from A_2 by a sequence of adjacent interchanges.

We will prove that these three possibilities are also equivalent for $\mathcal{A}_{m,n}(u, v|u', v')$. If A_1 can be obtained from A_2 by a sequence of adjacent interchanges, then we denote $A_2 \rightarrow A_1$.

Example 2.1. Let $u = (-1, 1, 1, 1, -1)$, $u' = (-1, -1, -1, -1, -1)$ and $v = v' = (-1, -1)$. Let

$$A_1 : \begin{array}{c|c|c|c|c|c|c|c} 0 & - & + & + & + & - & 0 & \\ \hline - & + & & & & & - & \\ \hline - & & & & & + & - & \\ \hline 0 & - & - & - & - & - & 0 & \end{array} \text{ and } A_2 : \begin{array}{c|c|c|c|c|c|c|c} 0 & - & + & + & + & - & 0 & \\ \hline - & & & & & + & - & \\ \hline - & + & & & & & - & \\ \hline 0 & - & - & - & - & - & 0 & \end{array}.$$

Since $\Sigma(A_1) \geq \Sigma(A_2)$, $A_1 \preceq_B A_2$ by definition (a). Also, A_1 can be obtained from A_2 by a sequence of adjacent interchanges such as;

$$\begin{array}{c}
A_2 : \begin{array}{c} 0 \parallel - \mid - \mid - \mid - \mid - \parallel 0 \\ \hline - \parallel \mid \mid \mid \mid \mid + \parallel - \\ \hline - \parallel + \mid \mid \mid \mid \mid \mid - \\ \hline 0 \parallel - \mid + \mid + \mid + \mid - \parallel 0 \end{array} \rightarrow \begin{array}{c} 0 \parallel - \mid - \mid - \mid - \mid - \parallel 0 \\ \hline - \parallel + \mid - \mid \mid \mid + \parallel - \\ \hline - \parallel \mid + \mid \mid \mid \mid \mid \mid - \\ \hline 0 \parallel - \mid + \mid + \mid + \mid - \parallel 0 \\ \hline 0 \parallel - \mid - \mid - \mid - \mid - \parallel 0 \\ \hline - \parallel + \mid \mid - \mid \mid + \parallel - \\ \hline - \parallel \mid \mid + \mid \mid \mid \mid \mid \mid - \\ \hline 0 \parallel - \mid + \mid + \mid + \mid - \parallel 0 \\ \hline 0 \parallel - \mid - \mid - \mid - \mid - \parallel 0 \\ \hline - \parallel + \mid \mid \mid - \mid + \parallel - \\ \hline - \parallel \mid \mid \mid + \mid \mid \mid \mid \mid \mid - \\ \hline 0 \parallel - \mid + \mid + \mid + \mid - \parallel 0 \end{array} \\
\rightarrow \begin{array}{c} 0 \parallel - \mid - \mid - \mid - \mid - \parallel 0 \\ \hline - \parallel + \mid \mid \mid + \mid \mid \mid - \\ \hline - \parallel \mid \mid + \mid \mid \mid \mid \mid \mid - \\ \hline 0 \parallel - \mid + \mid + \mid + \mid - \parallel 0 \\ \hline 0 \parallel - \mid - \mid - \mid - \mid - \parallel 0 \\ \hline - \parallel + \mid \mid \mid - \mid + \parallel - \\ \hline - \parallel \mid \mid \mid + \mid \mid \mid \mid \mid \mid - \\ \hline 0 \parallel - \mid + \mid + \mid + \mid - \parallel 0 \\ \hline 0 \parallel - \mid - \mid - \mid - \mid - \parallel 0 \\ \hline - \parallel + \mid \mid \mid \mid \mid \mid + \parallel - \\ \hline - \parallel \mid \mid \mid \mid \mid + \mid \mid \mid \mid \mid \mid - \\ \hline 0 \parallel - \mid + \mid + \mid + \mid - \parallel 0 \end{array} : A_1
\end{array}$$

Theorem 2.2. Let $u = (u_1, u_2, \dots, u_n), u' = (u'_1, u'_2, \dots, u'_n), v = (v_1, v_2, \dots, v_m)$, and $v' = (v'_1, v'_2, \dots, v'_m)$ be vectors of ± 1 's. Let A_1 and A_2 be in $\mathcal{A}_{m,n}(u, v|u', v')$. Then there exists unique $M \in \mathcal{A}_{m,n}(u, v|u', v')$ such that

$$\Sigma(M) = \max\{\Sigma(A_1), \Sigma(A_2)\} \text{ (entrywise maximum).}$$

Moreover, M can be obtained from each of A_1 and A_2 by a sequence of adjacent interchanges, and hence $M \preceq_B A_1, A_2$.

Proof. Let $A_1 = [a_{ij}^1], A_2 = [a_{ij}^2] \in \mathcal{A}_{m,n}(u, u'|v, v')$, and let $D = [d_{ij}] = \Sigma(A_1) - \Sigma(A_2)$, an integral matrix with last row and last column all 0's. If $D = 0$, then $A_1 = A_2$ and there is nothing to prove. So assume that $D \neq 0$.

If $u_j = +1$ then, since the ± 1 's alternate, the partial sum of column j of a matrix in $\mathcal{A}_{m,n}(u, u'|v, v')$ down to a row i is either 0 or -1 ; if $u_j = -1$, this partial sum is either 0 or 1. Let $k \geq 1$ be the smallest integer such that row k of D is nonzero. Then the following k -partial column sum property holds: *The partial column sums of A_1 and A_2 down to row $(k-1)$ are equal.* It thus follows that row k of D is a $(0, \pm 1)$ -vector. Let $l_1 \geq 1$ be the smallest integer such that $d_{kl_1} \neq 0$. Then $d_{kl_1} = \pm 1$, and since we may interchange A_1 and A_2 , we may assume that $d_{kl_1} = +1$. Let $l_2 \geq l_1$ be the smallest integer such that $d_{kj} = +1$ for $l_1 \leq j \leq l_2$ and $d_{k, l_2+1} \neq +1$.

Since $d_{kn} = 0$, such an l_2 exists. The k -partial column sum property implies that $d_{k,l_2+1} = 0$. Thus we have either

- Case (i) $a_{kl_1}^1 = a_{k,l_2+1}^2 = +1$ and $a_{kl_1}^2 = a_{k,l_2+1}^1 = 0$, or
 Case (ii) $a_{kl_1}^1 = a_{k,l_2+1}^2 = 0$ and $a_{kl_1}^2 = a_{k,l_2+1}^1 = -1$.

We only argue the first case (i) with the argument for the second case (ii) being very similar. So assume (i) holds.

Consider the sequences determined by columns l_1 and l_2 of A_2 :

$$u_{l_1}, a_{1l_1}^2, a_{2l_1}^2, \dots, a_{k-1,l_1}^2, (a_{kl_1}^2 + 1)$$

and

$$u_{l_2+1}, a_{1,l_2+1}^2, a_{2,l_2+1}^2, \dots, a_{k-1,l_2+1}^2, (a_{k,l_2+1}^2 - 1).$$

Since $a_{kl_1}^1 = 1$ and $a_{k,l_1}^2 = 0$, the $+1$'s and -1 's in the first of these sequences alternate. Since $a_{k,l_2+1}^1 = 0$ and $a_{k,l_2+1}^2 = +1$, the $+1$'s and -1 's in the second of these sequences alternate. Thus there exists a q with $l_1 \leq q \leq l_2$ such that in the sequences

$$u_q, a_{1q}^2, a_{2q}^2, \dots, a_{k-1,q}^2, (a_{kq}^2 + 1) \text{ and } u_{q+1}, a_{1,q+1}^2, a_{2,q+1}^2, \dots, a_{k-1,q+1}^2, (a_{k,q+1}^2 - 1)$$

the $+1$'s and -1 's alternate. We choose the smallest such q .

Now consider the alternating sequences

$$(2.1) \quad v_i, a_{i1}^2, a_{i2}^2, \dots, a_{iq}^2 \text{ for } i > k.$$

Suppose that the last nonzero entry each of these sequences is -1 for all $i > k$. Then for each such i we compute

$$\begin{aligned} d_{iq} &= \Sigma(A_1)_{iq} - \Sigma(A_2)_{iq} \\ &= \left(\Sigma(A_1)_{i-1,q} + \sum_{j=1}^q a_{ij}^1 \right) - \left(\Sigma(A_2)_{i-1,q} + \sum_{j=1}^q a_{ij}^2 \right) \\ &= (\Sigma(A_1)_{i-1,q} - \Sigma(A_2)_{i-1,q}) + \left(\sum_{j=1}^q a_{ij}^1 - \sum_{j=1}^q a_{ij}^2 \right) \\ &= d_{i-1,q} + \sum_{j=1}^q a_{ij}^1 - \sum_{j=1}^q a_{ij}^2. \end{aligned}$$

Since the last nonzero of each of the sequences (2.1) is -1 , we have that

$$\sum_{j=1}^q a_{ij}^1 \geq \sum_{j=1}^q a_{ij}^2.$$

Therefore $d_{iq} \geq d_{i-1,q}$ and so the sequence $d_{kq}, d_{k+1,q}, \dots, d_{nq}$ is nondecreasing. This contradicts the fact that $d_{kq} = 1$ and $d_{nq} = 0$. Let $p(\geq k)$ be the smallest integer such that the last nonzero of the sequence $v_{p+1}, a_{p+1,1}^2, a_{p+1,2}^2, \dots, a_{p+1,q}^2$ is $+1$ and let $A_2^1 = A_2 + T_{p,q}$, where $T_{p,q}$ is the matrix which is all zeros except for its 2×2 submatrix, determined by rows p and $p+1$ and columns q and $q+1$, equal to

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then $A_2^1 \in \mathcal{A}_{m,n}(u, v|u', v')$ such that $D - (\Sigma(A_1) - \Sigma(A_2^1))$ has just one nonzero entry $+1$.

As already indicated, Case (ii) proceeds in a very similar way. Thus continuing, we obtain for some k_1 and k_2 , that there are adjacent interchanges $A_1 \rightarrow A_1^1 \rightarrow A_1^2 \rightarrow \dots A_1^{k_1}$ and $A_2 \rightarrow A_2^1 \rightarrow A_2^2 \rightarrow \dots A_2^{k_2}$, with $A_1^{k_1} = A_2^{k_2}$ and $\Sigma(A_1^{k_1}) = \Sigma(A_2^{k_2}) = \max\{\Sigma(A_1), \Sigma(A_2)\}$ (entrywise). $\square$

Example 2.3. Let A and B be determined, respectively, by:

$$\begin{array}{c} \begin{array}{c|c|c|c|c|c|c|c} 0 & - & - & - & - & - & - & 0 \\ \hline - & & & & & + & & - \\ \hline + & & & & & & & - \\ \hline - & & & + & - & & & + \\ \hline - & + & & - & + & & & - \\ \hline + & & & & - & + & & - \\ \hline 0 & - & + & + & + & - & & 0 \end{array} \end{array} \quad \text{and} \quad \begin{array}{c} \begin{array}{c|c|c|c|c|c|c|c} 0 & - & - & - & - & - & - & 0 \\ \hline - & + & & & & & & - \\ \hline + & - & & & & & + & - \\ \hline - & + & & & & & - & + \\ \hline - & & & & & & + & - \\ \hline + & & & & & & & - \\ \hline 0 & - & + & + & + & - & & 0 \end{array} \end{array}.$$

Then we have

$$\Sigma(A) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}, \quad \Sigma(B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix},$$

and

$$D = \Sigma(A) - \Sigma(B) = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

First, we find the first nonzero row in D :

$$D = \left[\begin{array}{c|c|c|c|c} -1 & -1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Then for $k = 1$ we can find $q = 3$ and $p = 1$ in A .

$$\begin{array}{c|c|c|c|c|c|c} 0 & - & - & - & - & - & 0 \\ \hline - & & & & + & & - \\ \hline + & & & & & & - \\ \hline - & & & + & - & & + \\ \hline - & + & & - & + & & - \\ \hline + & & & & - & + & - \\ \hline 0 & - & + & + & + & - & 0 \end{array}.$$

Then we have $A + T_{1,3} = A_1 \in \mathcal{A}_{5,5}(u, v|u', v')$ such as

$$\begin{array}{c|c|c|c|c|c|c} 0 & - & - & - & - & - & 0 \\ \hline - & & & & + & & - \\ \hline + & & & & & & - \\ \hline - & & & + & - & & + \\ \hline - & + & & - & + & & - \\ \hline + & & & & - & + & - \\ \hline 0 & - & + & + & + & - & 0 \end{array} + \begin{array}{c|c|c|c|c|c|c} 0 & - & - & - & - & - & 0 \\ \hline - & & & + & - & & - \\ \hline + & & & - & + & & - \\ \hline - & & & & & & + \\ \hline - & & & & & & - \\ \hline + & & & & & & - \\ \hline 0 & - & + & + & + & - & 0 \\ \hline 0 & - & - & - & - & - & 0 \\ \hline - & & & + & & & - \\ \hline + & & & - & + & & - \\ \hline - & & & + & - & & + \\ \hline - & + & & - & + & & - \\ \hline + & & & & - & + & - \\ \hline 0 & - & + & + & + & - & 0 \end{array} = \begin{array}{c|c|c|c|c|c|c} 0 & - & - & - & - & - & 0 \\ \hline - & & & + & - & & - \\ \hline + & & & - & + & & - \\ \hline - & & & + & - & & + \\ \hline - & + & & - & + & & - \\ \hline + & & & & - & + & - \\ \hline 0 & - & + & + & + & - & 0 \end{array}.$$

If we denote $A + T_{p,q} = A_1$ by $A \xrightarrow{(p,q)} A_1$, then we have

$$A \xrightarrow{(1,3)} A_1 \xrightarrow{(1,2)} A_2 \xrightarrow{(1,1)} A_3,$$

$$B \xrightarrow{(2,4)} B_1,$$

$$A_3 \xrightarrow{(3,2)} A_4 \xrightarrow{(3,1)} A_5,$$

and

$$B_1 \xrightarrow{(4,4)} B_2,$$

where $A_5 = B_2$ such as

0		-		-		-		-		-		0
-		+										-
+		-						+				-
-		+						-				+
-								+				-
+								-		+		-
0		-		+		+		+		-		0

Therefore we have $A \vee B = A_5 = B_2$.

Corollary 2.4. *The three possibilities (a), (b), and (c) are equivalent for $\mathcal{A}_{m,n}(u, v|u', v')$ and each defines the Bruhat order $\preceq_B$ on $\mathcal{A}_{m,n}(u, v|u', v')$.*

Proof. Let $A_1, A_2 \in \mathcal{A}_{m,n}(u, v|u', v')$ with $\Sigma(A_1) \geq \Sigma(A_2)$. Then $\max\{\Sigma(A_1), \Sigma(A_2)\} = \Sigma(A_1)$, and by Theorem 2.2 A_1 can be obtained from A_2 by a sequence of adjacent interchanges. The corollary now follows from the discussion preceding Theorem 2.2. $\square$

Note that this corollary implies that, as with ordinary ASMs, there is no difference between the Bruhat order and the weak Bruhat order on $\mathcal{A}_{m,n}(u, v|u', v')$.

Corollary 2.5. *The partially ordered set $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ is a distributive lattice where for $A_1, A_2 \in \mathcal{A}_{m,n}(u, v|u', v')$, the meet and join are given by*

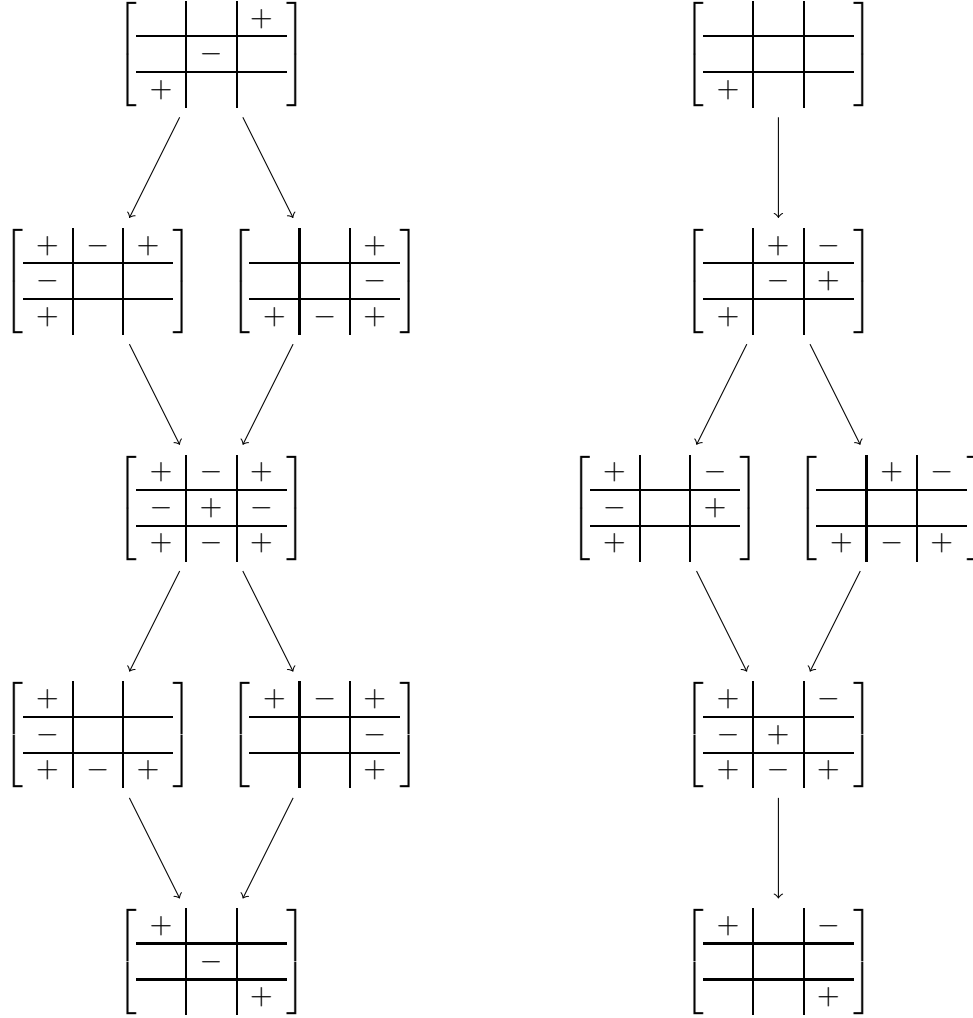
- (i) $A_1 \wedge A_2 = B$ where $B \in \mathcal{A}_{m,n}(u, v|u', v')$ such that $\Sigma(B) = \max\{\Sigma(A_1), \Sigma(A_2)\}$.
- (ii) $A_1 \vee A_2 = C$ where $C \in \mathcal{A}_{m,n}(u, v|u', v')$ such that $\Sigma(C) = \min\{\Sigma(A_1), \Sigma(A_2)\}$.

Proof. By Theorem 2.2 $\mathcal{A}_{m,n}(u, v|u', v')$ has a well-defined meet and assertion (i) follows. Assertion (ii) can be obtained from the analogue of Theorem 2.2 with minimum replacing maximum; it also follows from the fact that $\mathcal{A}_{m,n}(u, v|u', v')$ is finite and so the existence of joins follows from the existence of meets.

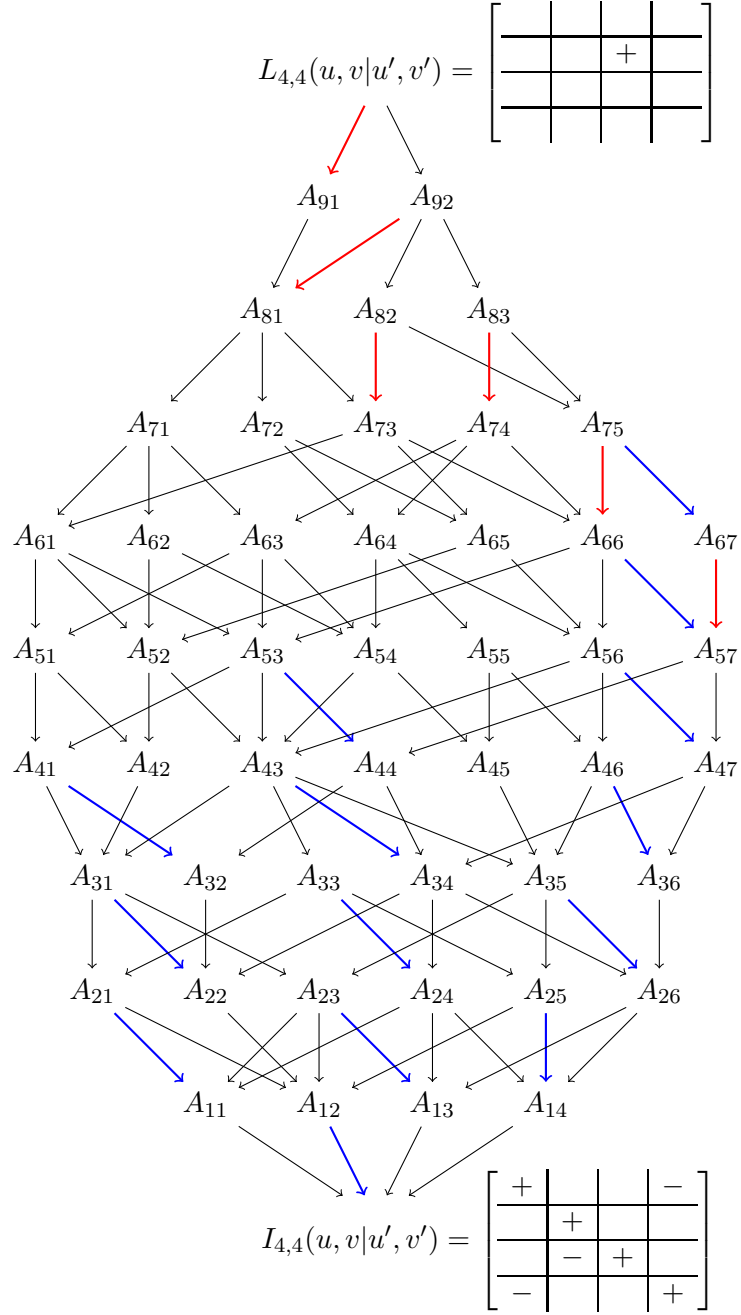
In order that $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ be distributive, we must have

$$A_1 \wedge (A_2 \vee A_3) = (A_1 \vee A_2) \wedge (A_1 \vee A_3)$$

for all $A_1, A_2, A_3 \in \mathcal{A}_{m,n}(u, v|u', v')$. The distributive property for the real numbers with the usual $\leq$ order relation holds trivially. It then follows from (i) and (ii) that $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ is also distributive. $\square$

Figure 1: Hasse diagrams of $\mathcal{K}_1$ and $\mathcal{K}_2$.

In [8], we have the (u, v) -identity matrix $I_n(u, v)$ is the unique minimal element in $\mathcal{A}_n(u, v)$, and the (u, v) -anti-identity matrix $L_n(u, v)$ is the unique maximal element in $\mathcal{A}_n(u, v)$. We know that there is the unique minimal element $I_{m,n}(u, v|u', v')$ and the unique maximal element $L_{m,n}(u, v|u', v')$ in $\mathcal{A}_{m,n}(u, v|u', v')$, but do not characterize $I_{m,n}(u, v|u', v')$ and $L_{m,n}(u, v|u', v')$ yet. For a matrix $A \in \mathcal{A}_{m,n}(u, v|u', v')$, let $\rho'(A)$ equal the sum of the entries of $\Sigma(A)$, and let $\rho(A) = \rho'(I_{m,n}(u, v|u', v')) - \rho'(A)$.

Figure 2: Hasse diagram of $(\mathcal{A}_{4,4}(u, v|u', v'), \preceq_B)$

Corollary 2.6. *The partially ordered set $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ is a graded lattice where the rank of $A \in (\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ equals $\rho(A)$.*

Proof. By Corollary 2.5 $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ is a lattice. Let $A_1, A_2 \in \mathcal{A}_{m,n}(u, v|u', v')$ where A_2 covers A_1 . Then $\Sigma(A_1) - \Sigma(A_2) \geq 0$ and $A_1 \wedge A_2 = A_1$. By Theorem 2.2, A_1 can be obtained from A_2 by a sequence of adjacent interchanges. Each interchange increases the sum of the entries of the corresponding Σ -matrix by 1 and gives a matrix $A' \in \mathcal{A}_{m,n}(u, v|u', v')$ with $A_1 \preceq_B A' \preceq_B A_2$. Since A_2 covers A_1 , we have that $A_1 = A'$ and $\rho'(A_1) = \rho'(A_2) + 1$. Hence A_1 can be obtained from A_2 by one adjacent interchange and $\rho(A_1) = \rho(A_2) - 1$. Hence $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ is graded with rank function $\rho(\cdot)$ where $\rho(I_{m,n}(u, v|u', v')) = 0$ and $\rho(L_{m,n}(u, v|u', v')) = \rho'(I_{m,n}(u, v|u', v')) - \rho'(L_{m,n}(u, v|u', v'))$. $\square$

Example 2.7. Let

$$\mathcal{K}_1 = (\mathcal{A}_{3,3}(-1, +1, -1), \preceq_B)$$

and

$$\mathcal{K}_2 = (\mathcal{A}_{3,3}(u, v|u', v'), \preceq_B)$$

where

$$u = (-1, -1, +1), v = (-1, +1, -1), u' = (-1, +1, -1), v' = (+1, -1, -1).$$

Then we have

$$\rho(\mathcal{K}_1) = \rho(\mathcal{K}_2) = 4$$

and Hasse diagrams of $\mathcal{K}_1$ and $\mathcal{K}_2$ are Figure 1.

Hasse diagram of $\mathcal{K}_1$ is isomorphic to Hasse diagram of $(\mathcal{A}_3, \preceq_B)$ in [8], but Hasse diagram of $\mathcal{K}_2$ is not isomorphic.

Let

$$u = (-1, -1, -1, +1), v = (-1, -1, +1, +1), u' = (+1, +1, -1, -1)$$

and

$$v' = (+1, -1, -1, -1).$$

Then we have

$$(\mathcal{A}_{4,4}(u, v|u', v'), \preceq_B)$$

as Figure 2. Also we have

$$\rho(\mathcal{A}_{4,4}(u, v|u', v')) = 10 = \rho(\mathcal{A}_4)$$

and

$$|\mathcal{A}_{4,4}(u, v|u', v')| = 49 \neq 42 = |\mathcal{A}_4|.$$

3. RANK OF THE LATTICE $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$

Let $\rho(\cdot)$ denote the rank function of the graded lattice $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$. Let $L_{m,n}(u, v|u', v')$ and $I_{m,n}(u, v|u', v')$ the unique maximum element and the unique minimum element of this lattice, respectively. The rank of the lattice $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ equals $\rho(L_{m,n}(u, v|u', v'))$. Moreover, for $A \in \mathcal{A}_{m,n}(u, v|u', v')$, we have

$$\rho(A) = \rho'(I_{m,n}(u, v)) - \rho'(A) = \rho'(I_{m,n}(u, v) - A)$$

where recall that $\rho'(\cdot)$ denotes the sum of the entries of $\Sigma(\cdot)$. We have $\rho(\mathcal{A}_n) = \binom{n+1}{3}$, see e.g. [4].

Theorem 3.1. *Let $u = (u_1, u_2, \dots, u_n)$, $u' = (u'_1, u'_2, \dots, u'_n)$, $v = (v_1, v_2, \dots, v_m)$, and $v' = (v'_1, v'_2, \dots, v'_m)$ be vectors of ± 1 's. Suppose $m \leq n$. Then we have*

$$\rho(\mathcal{A}_{m,n}(u, v|u', v')) \leq \begin{cases} \frac{1}{12}(m-1)(m+1)(3n-m), & \text{if } m \text{ odd,} \\ \frac{1}{12}m(3mn-m^2-2), & \text{if } m \text{ even} \end{cases}.$$

Proof. Let $\rho'(A)$ equal the sum of the entries of $\Sigma(A)$ and $\rho(A) = \rho'(I_{m,n}(u, v|u', v')) - \rho'(A)$ for $A \in \mathcal{A}_{m,n}(u, v|u', v')$. Let $D(A) = [d_{ij}] = \Sigma(A) - \Sigma(I_{m,n}(u, v|u', v'))$. Then we have for each integer i, j such that $1 \leq i \leq m$ and $1 \leq j \leq n$,

- (1) $d_{ij} \geq 0$,
- (2) $d_{in} = d_{mj} = 0$,
- (3) $d_{i1} = d_{1j} \in \{0, 1\}$,
- (4) $|d_{ij} - d_{i,j+1}| \leq 1$,
- (5) $|d_{ij} - d_{i+1,j}| \leq 1$.

Therefore we have

$$D(A) \leq D^* = [d_{ij}^*] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 & 0 \\ 1 & 2 & 3 & \cdots & 3 & 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & 3 & 2 & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{array} \right].$$

Therefore $\rho(A) = \sum d_{ij} \leq \sum d_{ij}^*$.

Case (i) m is odd.

$$\rho(A) \leq \sum d_{ij}^* = (m-1)(n-1) + (m-3)(n-3) + \dots + 2 \cdot (n-m+2)$$

$$\begin{aligned}
&= \sum_{k=1}^{\frac{m-1}{2}} 2k \cdot (n - m + 2k) \\
&= \sum_{k=1}^{\frac{m-1}{2}} \{4k^2 + 2k(n - m)\} \\
&= \frac{(m^2 - 1)m}{6} + \frac{(m^2 - 1)(n - m)}{4} \\
&= \frac{m^2 - 1}{12}(3m - m) = \frac{1}{12}(m - 1)(m + 1)(3n - m)
\end{aligned}$$

Case (ii) m is even.

$$\begin{aligned}
\rho(A) \leq \sum d_{ij}^* &= (m - 1)(n - 1) + (m - 3)(n - 3) + \dots + 1 \cdot (n - m + 1) \\
&= \sum_{k=1}^{\frac{m}{2}} (2k - 1)(n - m + 2k - 1) \\
&= \sum_{k=1}^{\frac{m}{2}} \{4k^2 + 2k(n - m - 2) + (-n + m + 1)\} \\
&= \frac{m(m + 2)(m + 1)}{6} + \frac{m(m + 2)(n - m - 3)}{4} + \frac{m(-n + m + 1)}{2} \\
&= \frac{m}{12}(3mn - m^2 - 2)
\end{aligned}$$

Therefore we have the result. $\square$

Corollary 3.2. *Let $u = (u_1, u_2, \dots, u_n)$, $u' = (u'_1, u'_2, \dots, u'_n)$, $v = (v_1, v_2, \dots, v_n)$, and $v' = (v'_1, v'_2, \dots, v'_n)$ be vectors of ± 1 's. Then we have*

$$\rho(\mathcal{A}_{n,n}(u, v|u', v')) \leq \binom{n+1}{3}.$$

Proof. If n is odd, then we have by Theorem 3.1,

$$\rho(\mathcal{A}_{n,n}(u, v|u', v')) \leq \frac{1}{12}(n - 1)(n + 1)(3n - n) = \frac{(n - 1)n(n - 2)}{6} = \binom{n+1}{3}.$$

If n is even, then we have by Theorem 3.1,

$$\rho(\mathcal{A}_{n,n}(u, v|u', v')) \leq \frac{n}{12}(3n^2 - n^2 - 2) = \frac{(n - 1)n(n - 2)}{6} = \binom{n+1}{3}.$$

$\square$

4. CLOSING REMARKS

In this paper, we extend the result in [8] to a more general situation. We show another example of the lattice $(\mathcal{A}_{4,4}(u, v|u', v'), \preceq_B)$ for $u = v = (+1, -1, +1, -1)$ and $u' = v' = (-1, +1, -1, +1)$ in Figure 3.

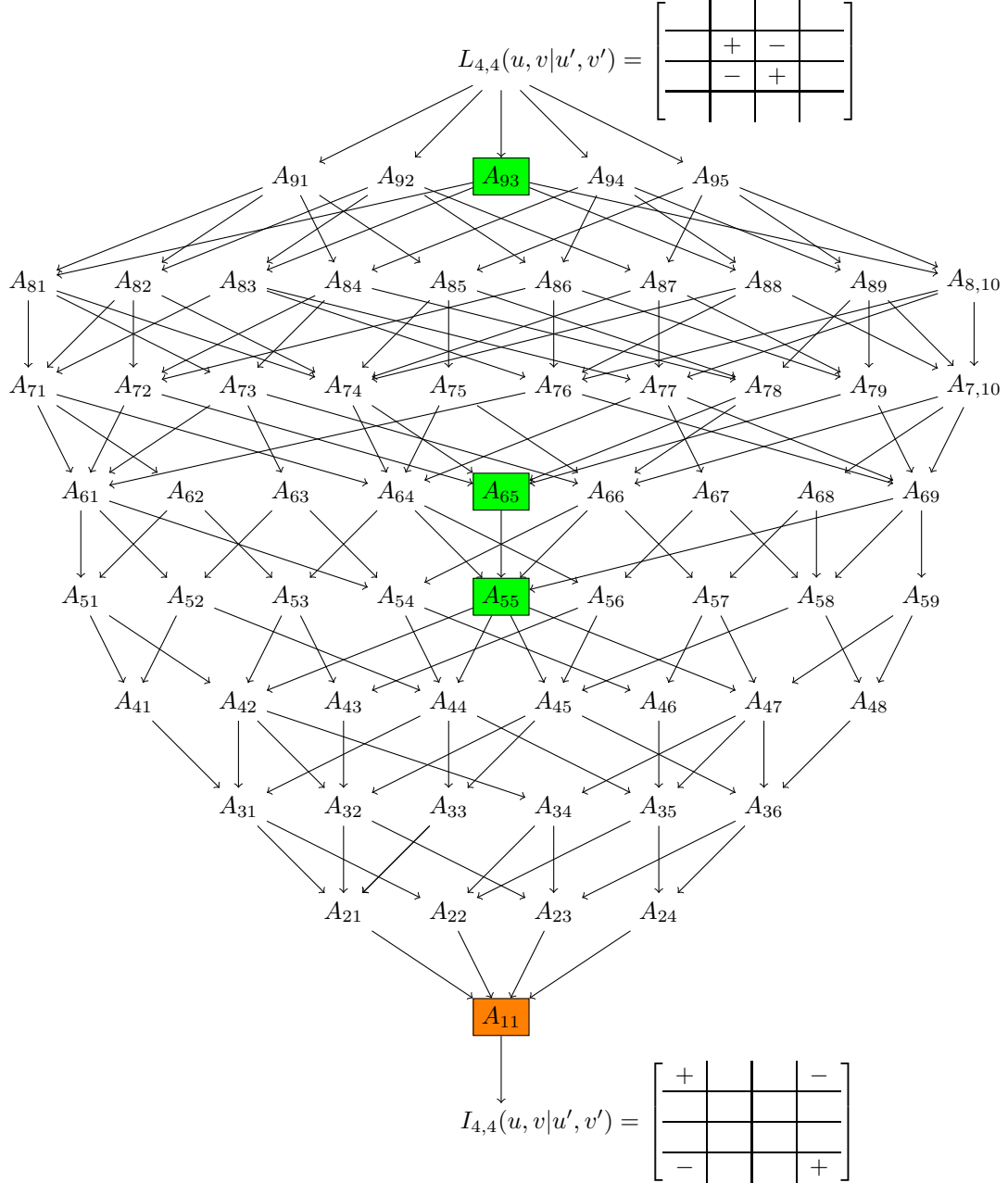


Figure 3: Hasse diagram of $(\mathcal{A}_{4,4}(u, v|u', v'), \preceq_B)$

A_{11} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td></td><td>-</td><td>+</td></tr></table>	+		-	-	+			-	+	A_{21} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td></td><td>-</td><td>+</td></tr></table>	+		-	-	+			-	+	A_{22} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td></td><td>-</td><td>+</td></tr></table>	+		-	-	+			-	+	A_{23} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td></td><td>-</td><td>+</td></tr></table>	+		-	-	+			-	+
+		-																																					
-	+																																						
	-	+																																					
+		-																																					
-	+																																						
	-	+																																					
+		-																																					
-	+																																						
	-	+																																					
+		-																																					
-	+																																						
	-	+																																					
A_{24} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td></td><td></td><td></td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+	-	+					+	-	A_{31} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+		-	-		+		+	-	A_{32} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td>+</td><td>-</td><td></td></tr></table>	+		-	-		+	+	-		A_{33} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td></td><td></td><td></td></tr><tr><td></td><td></td><td></td></tr></table>	+	-	+						
+	-	+																																					
	+	-																																					
+		-																																					
-		+																																					
	+	-																																					
+		-																																					
-		+																																					
+	-																																						
+	-	+																																					
A_{34} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td></td><td>-</td><td></td></tr></table>	+		-	-		+		-		A_{35} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td></td><td></td><td>-</td></tr><tr><td></td><td>+</td><td></td></tr></table>	+	-	+			-		+		A_{36} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td>-</td><td></td><td>-</td></tr></table>	+	-	+	-	+		-		-	A_{41} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td></td><td></td><td></td></tr></table>	+		-		-	+			
+		-																																					
-		+																																					
	-																																						
+	-	+																																					
		-																																					
	+																																						
+	-	+																																					
-	+																																						
-		-																																					
+		-																																					
	-	+																																					
A_{42} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td>+</td><td>-</td><td></td></tr></table>	+		-	-		+	+	-		A_{43} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td></td><td></td><td>-</td></tr></table>	+		-	-		+			-	A_{44} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td>-</td></tr></table>	+	-	+	-		-	-	+	-	A_{45} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td>+</td><td>-</td><td></td></tr></table>	+	-	+	-	+		+	-	
+		-																																					
-		+																																					
+	-																																						
+		-																																					
-		+																																					
		-																																					
+	-	+																																					
-		-																																					
-	+	-																																					
+	-	+																																					
-	+																																						
+	-																																						
A_{46} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td></td><td>+</td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr></table>	+	-			+	-	-		+	A_{47} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td></tr><tr><td>+</td><td></td><td>-</td></tr></table>	+	-	+	-	+	-	+		-	A_{48} <table><tr><td></td><td></td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr></table>			+	+		-	-		+	A_{51} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td>+</td><td>-</td><td></td></tr></table>	+		-		-	+	+	-	
+	-																																						
	+	-																																					
-		+																																					
+	-	+																																					
-	+	-																																					
+		-																																					
		+																																					
+		-																																					
-		+																																					
+		-																																					
	-	+																																					
+	-																																						
A_{52} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td></td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td></td></tr></table>	+	-	+			-	-	+		A_{53} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+		-	-		+		+	-	A_{54} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td></td><td></td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td></tr></table>	+	-				+	-	+	-	A_{55} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td></tr><tr><td>+</td><td>-</td><td>+</td></tr></table>	+	-	+	-	+	-	+	-	+
+	-	+																																					
		-																																					
-	+																																						
+		-																																					
-		+																																					
	+	-																																					
+	-																																						
		+																																					
-	+	-																																					
+	-	+																																					
-	+	-																																					
+	-	+																																					
A_{56} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td></td><td></td><td>-</td></tr></table>	+	-	+	-	+				-	A_{57} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td>+</td><td></td><td>-</td></tr></table>	+	-		-	+		+		-	A_{58} <table><tr><td></td><td></td><td>+</td></tr><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td>+</td><td>-</td></tr></table>			+	+	-		-	+	-	A_{59} <table><tr><td></td><td></td><td>+</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr></table>			+		-	+	+		-
+	-	+																																					
-	+																																						
		-																																					
+	-																																						
-	+																																						
+		-																																					
		+																																					
+	-																																						
-	+	-																																					
		+																																					
	-	+																																					
+		-																																					
A_{61} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td></tr><tr><td>+</td><td></td><td></td></tr></table>	+	-	+	-	+	-	+			A_{62} <table><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td></td><td></td><td></td></tr></table>	+		-	-		+				A_{63} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td></td><td></td><td></td></tr><tr><td>+</td><td>-</td><td></td></tr></table>	+	-					+	-		A_{64} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>+</td><td>+</td><td>-</td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+	-	+	+	+	-		+	-
+	-	+																																					
-	+	-																																					
+																																							
+		-																																					
-		+																																					
+	-																																						
+	-																																						
+	-	+																																					
+	+	-																																					
	+	-																																					
A_{65} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr></table>	+	-	+	-		+	+		-	A_{66} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td>+</td><td>-</td><td>-</td></tr></table>	+	-		-	+		+	-	-	A_{67} <table><tr><td></td><td></td><td>+</td></tr><tr><td></td><td></td><td>-</td></tr><tr><td></td><td>-</td><td>+</td></tr></table>			+			-		-	+	A_{68} <table><tr><td></td><td></td><td></td></tr><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr></table>				+		-	-		+
+	-	+																																					
-		+																																					
+		-																																					
+	-																																						
-	+																																						
+	-	-																																					
		+																																					
		-																																					
	-	+																																					
+		-																																					
-		+																																					
A_{69} <table><tr><td></td><td></td><td>+</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td>+</td><td>-</td><td>-</td></tr></table>			+		-	+	+	-	-	A_{71} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td>-</td></tr><tr><td></td><td></td><td></td></tr></table>	+	-	+	-	+	-				A_{72} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td>+</td><td>-</td><td></td></tr></table>	+	-	+	-		+	+	-		A_{73} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td>+</td><td>-</td><td></td></tr></table>	+	-		-	+		+	-	
		+																																					
	-	+																																					
+	-	-																																					
+	-	+																																					
-	+	-																																					
+	-	+																																					
-		+																																					
+	-																																						
+	-																																						
-	+																																						
+	-																																						
A_{74} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+	-	+	-		+		+	-	A_{75} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+	-		-	+			+	-	A_{76} <table><tr><td></td><td></td><td>+</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td>+</td><td>-</td><td></td></tr></table>			+		-	+	+	-		A_{77} <table><tr><td></td><td></td><td>+</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td></td><td>+</td><td>-</td></tr></table>			+		-	+		+	-
+	-	+																																					
-		+																																					
	+	-																																					
+	-																																						
-	+																																						
	+	-																																					
		+																																					
	-	+																																					
+	-																																						
		+																																					
	-	+																																					
	+	-																																					
A_{78} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr></table>	+	-		-		+	+		-	A_{79} <table><tr><td></td><td>+</td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr></table>		+	-	-		+	+		-	$A_{7,10}$ <table><tr><td></td><td></td><td>+</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td>+</td><td>-</td><td>+</td></tr></table>			+		-	+	+	-	+	A_{81} <table><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td>+</td><td></td></tr><tr><td></td><td></td><td></td></tr></table>	+	-		-	+				
+	-																																						
-		+																																					
+		-																																					
	+	-																																					
-		+																																					
+		-																																					
		+																																					
	-	+																																					
+	-	+																																					
+	-																																						
-	+																																						
A_{82} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+	-	+	-		+		+	-	A_{83} <table><tr><td></td><td></td><td>+</td></tr><tr><td></td><td>-</td><td>+</td></tr><tr><td></td><td></td><td></td></tr></table>			+		-	+				A_{84} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td></td></tr></table>	+	-	+	+		-	-	+		A_{85} <table><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td></td><td>-</td></tr><tr><td></td><td>+</td><td>-</td></tr></table>	+	-	+	-		-		+	-
+	-	+																																					
-		+																																					
	+	-																																					
		+																																					
	-	+																																					
+	-	+																																					
+		-																																					
-	+																																						
+	-	+																																					
-		-																																					
	+	-																																					
A_{86} <table><tr><td></td><td>+</td><td>-</td></tr><tr><td>+</td><td>-</td><td>+</td></tr><tr><td>-</td><td>+</td><td></td></tr></table>		+	-	+	-	+	-	+		A_{87} <table><tr><td></td><td>+</td><td>-</td></tr><tr><td>-</td><td></td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr></table>		+	-	-		+	+		-	A_{88} <table><tr><td></td><td></td><td>+</td></tr><tr><td>+</td><td>-</td><td></td></tr><tr><td>-</td><td>+</td><td></td></tr></table>			+	+	-		-	+		A_{89} <table><tr><td></td><td>-</td><td>+</td></tr><tr><td>+</td><td></td><td>-</td></tr><tr><td>-</td><td>+</td><td>-</td></tr></table>		-	+	+		-	-	+	-
	+	-																																					
+	-	+																																					
-	+																																						
	+	-																																					
-		+																																					
+		-																																					
		+																																					
+	-																																						
-	+																																						
	-	+																																					
+		-																																					
-	+	-																																					

$A_{8,10}$	A_{91}	A_{92}	A_{93}
$\begin{bmatrix} & & & \\ & & & \\ & & + & - \\ & & - & + \end{bmatrix}$	$\begin{bmatrix} + & - & & \\ - & & + & \\ & + & - & \\ & & & \end{bmatrix}$	$\begin{bmatrix} & & + & - \\ & - & & + \\ & + & - & \\ & & & \end{bmatrix}$	$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$
A_{94}	A_{95}		
$\begin{bmatrix} & - & + & \\ + & & - & \\ - & + & & \end{bmatrix}$	$\begin{bmatrix} & - & + & \\ & + & - & - \\ & & - & + \end{bmatrix}$		

REFERENCES

1. D.M. Bressoud: Proofs and Confirmations; The Story of the Alternating Sign Matrix Conjecture. Math. Assoc. America, Washington, D.C., 1999. <https://doi.org/10.1017/CB09780511613449.002>
2. A. Björner & F. Brenti: Combinatorics of Coxeter Groups. Graduate Texts in Math. #231, Springer (2003). <https://doi.org/10.1007/3-540-27596-7>
3. R.A. Brualdi, K.P. Kiernan, S.A. Meyer & M.W. Schroeder: Patterns of alternating sign matrices. Linear Algebra Appl. **438** (2013), 3967-3990. <https://doi.org/10.1016/j.laa.2012.03.009>
4. R.A. Brualdi & M.W. Schroeder: Alternating sign matrices and their Bruhat order. Discrete Math. **340** (2017), 1996-2019. <https://doi.org/10.1016/j.disc.2016.10.010>
5. R.A. Brualdi & G. Dahl: Alternating sign matrices, extensions, and related cones. Adv. in Appl. Math. **86** (2017), 19-49. <https://doi.org/10.1016/j.aam.2016.12.001>
6. R.A. Brualdi & G. Dahl: Alternating sign matrices: Extensions, König-properties, and primary sum-sequences. Graph and Combinatorics **36** (2020), 63-92. <https://doi.org/10.1007/s00373-019-02119-x>
7. R.A. Brualdi & H.K. Kim: A generalization of alternating sign matrices. J. Combin. Des. **23** (2015), 204-215. <https://doi.org/10.1002/jcd.21397>
8. R.A. Brualdi & H.K. Kim: A generalized alternating sign matrices and signed permutation matrices. J. Korean Math. Soc. **58** (2021), no. 4, 921-948. <https://doi.org/10.4134/JKMS.j200318>
9. R.A. Brualdi & H.K. Kim: Completions of alternating sign matrices. Graphs and Combinatorics **31** (2015), no. 3, 507-522. <https://doi.org/10.1007/s00373-014-1409-1>
10. A. Lascoux & M. Schützenberger: Treillis et bases des groupes de Coxeter. (French) [Lattices and bases of Coxeter groups] Electron. J. Combin. **3** (1996), no. 2, Research paper 27. <https://doi.org/10.37236/1285>

PROFESSOR: DEPARTMENT OF MATHEMATICS EDUCATION, SANGMYUNG UNIVERSITY, SEOUL 03016, REPUBLIC OF KOREA
Email address: indices@smu.ac.kr