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THE EXISTENCE OF A SOLUTION OF THE INTEGRAL EQUATION ON TRIPLED b-METRIC SPACES

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Abstract. In this paper, we introduce a tripled b-metric spaces and tripled Hardy-Rogers-type (F, β)-contraction and establish fixed point theorems for these contractions and use our fixed point theorems to prove the existence theorem for Volterra-type integral inclusion.

1. Introduction

The theory of differential equations are based on nonlinear functional analysis. Many existence theorems for the solution of differential equations are proved by means of fixed point theorems. The famous Banach contraction principle has a lot of applications in theory of integral equations. There are many generalizations of Banach contraction principle, see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36], Wardowski [37] gave an interesting generalization of Banach contraction known as F-contraction. Several authors generalized F-contraction by combining it with some existing contractive conditions, see, for example, Acar and Altun [1], Batra and Vashistha [6], Cosentino and Vetro [13], Mınak et al. [22], Paesano and Vetro [26], Piri and Kumam [29], Secelean [31], and Sgroi and Vetro [32].

The problem of the convergence of measurable functions with respect to a measure, lead to a generalization of notion of a metric. Using this idea, Czerwik [14] gave a generalization of the famous Banach fixed point theorem [14] in so-called b-metric spaces. For some important results on b-metric spaces, we refer the reader to [4, 9, 10, 15, 33].

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In 2014, Cosentino et al. [12] extended F-contraction in the setting of b-metric spaces and proved some fixed point theorems. In this article, we generalize the result of Cosentino et al. for new class of F-contractions in the setting of b-metric spaces.We also construct an example to show the generality of our result. Finally, we apply our result to get existence theorems for Volterra-type integral inclusion in tripled b-metric spaces.

2. Preliminaries

In this section we explain the following definitions, notions and results.

Definition 2.1. [14] Let Y be a nonempty set. A mapping $d : Y \times Y \to [0, 1)$ is said to be a b-metric on Y if for each $x, y, z \in Y$, we have a real number $s > 1$ such that

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then the triplet (Y, d, s) is said to be a b-metric space.

Note that every metric space is a b-metric but converse is not true.

Example 2.2. Let $Y = [0, 1)$ and $d : Y \times Y \to [0, 1)$, $\phi(x, y) = |x - y|^2$ for each $x, y \in Y$. Obviousaly, $(Y, \phi, 2)$ is a b-metric space, but not a metric space.

Lemma 2.3. [14] *Let (Y, ϕ, s) be a b-metric space, and suppose $\{y_n\}$ be a sequence in Y. If $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} y_n = z$, then $y = z$.*

Let (Y, d, s) be a b-metric space. The closed and bounded sets in Y are defined in a similar for a metric space. We denote by $CB(Y)$ the class of all nonempty closed and bounded subsets of Y and by $CL(Y)$ the class of all nonempty closed subsets of Y. Let $x \in Y$ and $A \subset Y$, $d(x, A) = \inf\{d(x, a) : a \in A\}$. For $A, B \in CB(Y)$, the function $H : CB(Y) \times CB(Y) \to [0, 1)$,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

is said to be a Hausdorff b-metric induced by the b-metric d [12]. For $A, B \in CL(Y)$, the function $H : CL(Y)\times CL(Y) \to [0, 1)$, given by $H(A, B) = \max\{\sup_{x\in A} d(x, B)$, $\sup_{y\in B} d(y, A)\}$, if the maximum exists and $H(A, B) = \infty$ otherwise, is said to be a generalized Hausdorff b-metric induced by b-metric d. Following properties based on b-metric are taken from [15].

Lemma 2.4. *Let (Y, ϕ, s) be a b-metric space. For any $A, B, C \in CB(Y)$ and any $x, y \in Y$, we have the following:*

(i) $d(x, A) \leq d(x, a)$ *for each* $a \in A$;
(ii) $d(x, B) \leq H(A, B)$ *for each* $x \in A$;
(iii) $H(A, A) = 0$;
(iv) $H(A, B) = H(B, A)$;
(v) $H(A, B) \leq s[H(A, C) + H(C, B)]$;
(vi) $d(x, A) \leq s[d(x, y) + d(y, A)]$.

Lemma 2.5. [16] *Let (Y, ϕ, s) be a b-metric space. For any $A, B \in CL(Y)$ and any $x \in Y$, we have the following:*

(i) *For $h > 1$ and $a \in B$, there exists $b \in C$ such that $d(a, b) \leq hH(B, C)$;*
(ii) $d(x, A) = 0$ *if and only if* $y \in \overline{B} = B$, *where* $\overline{B}$ *denotes the closure of the set* B.

Definition 2.6. [12] Let $s > 1$ be a real number. Show by F_s the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, that is, for any $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$;
(F2) For each sequence $\{y_n\}$ of positive real numbers, we have $\lim_{n\to\infty} y_n = 0$ if and only if $\lim_{n\to\infty} F(y_n) = -\infty$;
(F3) For each sequence $\{y_n\}$ of positive real numbers with $\lim_{n\to\infty} y_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} y_n^k F(y_n) = 0$;
(F4) For each sequence $\{y_n\}$ of positive real numbers such that $\tau + F(sy_n) \leq F(y_{n-1})$, for each $n \in \mathbb{N}$ and some $\tau > 0$, we have $\tau + F(s^n y_n) \leq F(s^{n-1} y_{n-1})$ for each $n \in \mathbb{N}$.

Cosentino et al. also showed that the following functions belong to F_s [12].

- $F(x) = x + \ln x$ for each $x > 0$.
- $F(x) = \ln x$ for each $x > 0$.

We notice that G. Prasad and R.C. Dimri in 2018 proved for weakly contractive mappings using locally T-transitivity of binary relation and presenting a variant of Harjang and Sadarangani theorem involving more general relation theoretic metrical notions [42].

M. Dhanraj, A.J. Gnanapraksam, G. Mani. O. Ege, M.D.L. Sen proved fixed point theorem via orthogonal Geraphy type a-admisible contraction map in an orthogonal complete Branciari b-metric spaces context and provided an appication to find the existence and uniqueness of a solution to the Volterra integral equations [38].

3. MAIN RESULTS

Definition 3.1. Let Y is a nonempty set. A mapping $d_b : Y \times Y \times Y \to \mathbb{R}^+$ is told to be a tripled b-metric on Y, if for each $a, b, c \in Y$, we have a real number $s \geq 1$, such that

- (T_{b_1}) $d_b(a, b, c) = 0$ if and only if $a = b = c$;
- (T_{b_2}) $d_b(a, b, c) > 0$ id and only if $a \neq b$ or $a \neq c$ or $b \neq c$;
- (T_{b_3}) $d_b(a, b, c) = d_b(a, c, b) = d_b(c, b, a) = d_b(b, a, c) = d_b(c, a, b) = d_b(b, c, a)$ for all $a, b, c \in Y$;
- (T_{b_4}) for all $a, b, c \in Y$,
  - (i) $d_b(a, a, b) = d_b(a, b, b)$;
  - (ii) $d_b(a, a, b), d_b(a, a, c), d_b(b, b, c) \leq d_b(a, b, c)$;
- (T_{b_5}) $d_b(a, b, c) \leq s\big[d_b(a, b, m) + d_b(m, b, c)\big]$ for all $a, b, c, m \in Y$,

Definition 3.2. Let (Y, d_b) be a tripled b-metric space, a sequence $\{y_n\}$ in Y is said to be

- (i) Cauchy sequence, if $\lim_{n,m\to\infty} d_b\left(y_n, y_m, y_m\right) = 0$;
- (ii) Convergent to a point $y \in Y$, if $\lim_{n\to\infty} d_b\left(y, y, y_n\right) = \lim_{n\to\infty} d_b\left(y, y_n, y_n\right) = 0$;
- (iii) (Y, d_b) is said to be complete, if every Cauchy sequence in (Y, d_b) convergent.

Definition 3.3. Let (Y, d_b) be a tripled b-metric space and let $\beta : Y \times Y \times Y \to [0, \infty)$ be a function.

- (i) A mapping $S : Y \to CL^b(Y)$ is β_s-admissible if for $x \in Y$, $y \in Sx$, $z \in Sy$ and $w \in Sz$ so $\beta(x, y, z) \geq s^2$, we have $\beta(y, z, w) \geq s^2$;
- (ii) A mapping $S : Y \to CL^b(Y)$ be $\beta_s^\star$-admissible mapping if for $x, y, z \in Y$ with $\beta(x, y, z) \geq s^2$, we have $\beta^\star(Sx, Sy, Sz) \geq s^2$, where

$$\beta^\star(Sx, Sy, Sz) = \inf\big\{\beta(u, v, w) : u \in Sx, v \in Sy, w \in Sz\big\}.$$

Example 3.4. Let $Y = \mathbb{Z}$ and we define $d_b : Y \times Y \times Y \to [0, \infty)$ as following

$$d_b(x, y, z) = \max\left\{|x-y|^2, |x-z|^2, |y-z|^2\right\}.$$

Then (Y, d_b) is a tripled b-metric space with $s = 2$. Define $\beta : Y \times Y \times Y \to [0, \infty)$ as follows

$$\beta(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ & \\ 6, & \begin{cases} x \neq y \text{ or } x \neq z \text{ or } z \neq y \\ \text{or } x \neq y, y \neq z, x \neq z. \end{cases} \end{cases}$$

We define $S : Y \to CL^b(Y)$ as follows

$$Sx = \begin{cases} \{-x\}, & x < 0, \\ \{1\}, & x = 0, \\ \{-x\}, & x > 0. \end{cases}$$

Now, we have the following cases.

Case I. If $x = 0$, $y \in \{1\}$, $z \in \{-1\}$ and $w \in \{1\}$, then we have $x = 0$, $y = 1$, $z = -1$ and $w = 1$. Thus $\beta(0, 1, -1) = 6 \geq s^2$ and $\beta(1, -1, 1) = 6 \geq s^2$.

Case II. If $x = -1$ then $y = 1$, $z = -1$ and $w = 1$. We have $\beta(-1, 1, -1) = 6 \geq s^2$ and $\beta(1, -1, 1) = 6 \geq s^2$.

Case III. If $x = 1$ then $y = -1$, $z = 1$ and $w = -1$. We have $\beta(0, -1, 1) = 6 \geq s^2$ and $\beta(-1, 1, -1) = 6 \geq s^2$. Then S is a β_s-admissible.

Case IV. If $x = 0$, $y = -1$, and $z = -1$, then we have $\beta(0, -1, -1) = 6 \geq s^2$. But $Tx = T0 = \{1\}$, $Sy = S(-1) = \{1\}$, $Sz = S(-1) = \{1\}$. If $u \in \{1\}$, $v \in \{1\}$, and $w \in \{1\}$ then we obtain $u = 1$, $v = 1$, $w = 1$, thus $\beta(1, 1, 1) = 0 \ngeq s^2$ and S is not a $\beta_s^\star$-admissible.

Lemma 3.5. *Let (Y, d_b) is a tripled b-metric space and $\{y_n\}$ be any sequence in Y, for the there exist $r > 0$ and $F \in \mathcal{F}$, such that*

$$r + F\left(sd_b\left(y_n, y_{n+1}, y_{n+2}\right) \leq F\left(d_b\left(x_{n-1}, y_n, y_{n+1}\right)\right)\right). \tag{1}$$

Then $\{y_n\}$ is a Cauchy sequence in Y.

Proof. Let $d_n = d_b\left(y_n, y_{n+1}, y_{n+1}\right)$ and $d_{n-1} = d_b\left(x_{n-1}, y_n, y_n\right)$ for any $n \in \mathbb{N}$. Then from (1) and property (F_4), we get $r + F(s^n d_n) \leq F(s^{n-1} d_{n-1})$, for all $n \in \mathbb{N}$. Consequently, we get

$$F\left(s^n d_n\right) \leq F\left(d_0\right) - nr. \tag{2}$$

For any $n \in \mathbb{N}$. By taking limit $n \to \infty$ in (2), we have $\lim_{n\to\infty} F\left(s^n d_n\right) = -\infty$. Then by property (F_2), we get $\lim_{n\to\infty} s^n d_n = 0$, by (F_3), there exist $k \in (0, 1)$ such that $\lim_{n\to\infty}(s^n d_n)^k F(s^n d_n) = 0$. From (2) we have

$$(s^n d_n)^k F(s^n d_n) - (s^n d_n)^k F(d_0) \leq -(s^n d_n)^k nr \leq 0. \tag{3}$$

For any $n \in \mathbb{N}$. By limit if $n \to \infty$ in (3), we get

$$\lim_{n\to\infty} n (s^n d_n)^k = 0. \tag{4}$$

By (4) that there exist $n_1 \in \mathbb{N}$ such that $n(s^n d_n)^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$s^n d_n \leq \frac{1}{n^{\frac{1}{k}}} \tag{5}$$

for $n \geq n_1$.

To prove that $\{y_n\}$ is a Cauchy, consider $n, m \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (5), we have

$$\begin{aligned}
d_b(y_n,\ y_m, y_m) &\leq s\left[d_b(y_n, y_{n+1}, y_m) + d_b(y_{n+1}, y_m, y_m)\right] \\
&\leq s\left[s\left[d_b(y_n, y_{n+1}, y_{n+2}) + d_b(y_{n+1}, y_{n+2}, y_m)\right.\right. \\
&\quad \left.\left. + d_b(y_{n+1}, y_{n+2}, y_m) + d_b(y_{n+2}, y_m, y_m)\right]\right] \\
&= s^2 d_b(y_n, y_{n+1}, y_{n+2}) + s^2\left[2s\left[d_b(y_{n+1}, y_{n+2}, y_{n+3}) + d_b(y_{n+2}, y_{n+3}, y_m)\right]\right] \\
&\quad + s^2\left[s\left(d_b(y_{n+2}, y_{n+3}, y_m) + d_b(y_{n+3}, y_m, y_m)\right)\right] \\
&= s^2 d_b(y_n, y_{n+1}, y_{n+2}) + 2s^3 d_b(y_{n+1}, y_{n+2}, y_{n+3}) + 2s^3 d_b(y_{n+2}, y_{n+3}, y_m) \\
&\quad + s^3 d_b(y_{n+2}, y_{n+3}, y_m) + s^3 d_b(y_{n+3}, y_m, y_m) \\
&\leq s^2 d_b(y_n, y_{n+1}, y_{n+2}) + 2s^3 d_b(y_{n+1}, y_{n+2}, y_{n+3}) \\
&\quad + 2s^3\left[s\left(d_b(y_{n+2}, y_{n+3}, y_{n+4}) + d_b(y_{n+3}, y_{n+4}, y_m)\right)\right] \\
&\quad + s^3\left[s\left(d_b(y_{n+2}, y_{n+3}, y_{n+4}) + d_b(y_{n+3}, y_{n+4}, y_m)\right)\right] \\
&\quad + s^4 d_b(y_{n+3}, y_{n+4}, y_m) + s^4 d_b(y_{n+4}, y_m, y_m) \\
&= s^2 d_b(y_n, y_{n+1}, y_{n+2}) + 2s^3 d_b(y_{n+1}, y_{n+2}, y_{n+3}) \\
&\quad + 3s^4 d_b(y_{n+2}, y_{n+3}, y_{n+4}) + 4s^4 d_b(y_{n+3}, y_{n+4}, y_m) + s^4 d_b(y_{n+4}, y_m, y_m) \\
&\leq s^2 d_b(y_n, y_{n+1}, y_{n+2}) + 2s^3 d_b(y_{n+1}, y_{n+2}, y_{n+3}) \\
&\quad + 3s^4 d_b(y_{n+2}, y_{n+3}, y_{n+4}) + 4s^4\left[s\left(d_b(y_{n+3}, y_{n+4}, y_{n+5}) + d_b(y_{n+4}, y_{n+5}, y_m)\right)\right] \\
&\quad + s^4\left[s\left(d_b(y_{n+4}, y_{n+5}, y_m) + d_b(y_{n+5}, y_m, y_m)\right)\right] \\
&= s^2 d_b(y_n, y_{n+1}, y_{n+2}) + 2s^3 d_b(y_{n+1}, y_{n+2}, y_{n+3}) + 3s^4 d_b(y_{n+2}, y_{n+3}, y_{n+4}) \\
&\quad + 4s^5 d_b(y_{n+3}, y_{n+4}, y_{n+5}) + 5s^5 d_b(y_{n+4}, y_{n+5}, y_m) + s^5 d_b(y_{n+5}, y_m, y_m).
\end{aligned}$$

Take $m = n + p$, we have

$$\begin{aligned} d_b\left(y_n, y_m, y_m\right) = {} & s^2 d_b\left(y_n, y_{n+1}, y_{n+2}\right) + 2s^3 d_b\left(y_{n+1}, y_{n+2}, y_{n+3}\right) \\ & + 3s^4 d_b\left(y_{n+2}, y_{n+3}, y_{n+4}\right) + 4s^5 d_b\left(y_{n+3}, y_{n+4}, y_{n+5}\right) \\ & + \ldots + (p-2)s^{p-1} d_b\left(y_{n+p-3}, y_{n+p-2}, y_{n+p-1}\right) \\ & + (p-1)s^p d_b\left(y_{n+p-2}, y_{n+p-1}, y_{n+p}\right) + s^{p-1} d_b\left(y_{n+p-1}, y_{n+p}, y_{n+p}\right). \end{aligned}$$

In a similar way, we have

$$\begin{aligned} d_b\left(y_n, y_m, y_m\right) \leq {} & s^2 \frac{1}{s^n n^{\frac{1}{k}}} + 2s^3 \frac{1}{s^{n+1}(n+1)^{\frac{1}{k}}} + 3s^4 \frac{1}{s^{n+2}(n+2)^{\frac{1}{k}}} \\ & + \ldots + (p-1)s^p \frac{1}{s^{n+p-2}(n+p-2)^{\frac{1}{k}}} \\ & + s^{p-1} d_b\left(y_{n+p-1}, y_{n+p}, y_{n+p}\right). \end{aligned}$$

Since
$d_b\left(y_n, y_{n+1}, y_{n+2}\right) \leq d_b\left(y_{n-1}, y_n, y_{n+1}\right)$ and $d_b\left(y_n, y_{n+1}, y_{n+2}\right) \leq \frac{1}{s} d_b\left(y_{n-1}, y_n, y_{n+1}\right)$, thus, we get

$$s^{p-1} d_b\left(y_{n+p-1}, y_{n+p}, y_{n+p}\right) \leq \frac{1}{s^n} d_b\left(x_0, y_1, y_1\right)$$

and

$$d_b\left(y_n, y_{n+p}, y_{n+p}\right) \leq s^{2-n}\left[\frac{1}{n^{\frac{1}{k}}} + \frac{2}{(n+1)^{\frac{1}{k}}} + \ldots + \frac{p-1}{(n+p-2)^{\frac{1}{k}}}\right] + \frac{1}{s^n} d_b\left(x_0, y_1, y_1\right).$$

Since $\frac{1}{n^{\frac{1}{k}}} + \frac{2}{(n+1)^{\frac{1}{k}}} + \ldots + \frac{p-1}{(n+p-2)^{\frac{1}{k}}}$ is convergent and $s^{2-n} \to 0$ as $n \to \infty$, thus

$$\lim_{n,p \to \infty} d_b\left(y_n, y_{n+p}, y_{n+p}\right) = 0.$$

This implies that $\{y_n\}$ is a Cauchy sequence. Now we define the notion of tripled-Hardy-Rogers- type (F, β)-contraction. ☐

Definition 3.6. Let (Y, d_b) be a tripled b-metric space and A, B and $C \in CL^b(Y)$, define $\delta_b(A, B, C)$ as follows

$$\begin{aligned} \delta_b(A, B, C) &= \sup\left\{\delta_b(a, B, C) : a \in A\right\}, \\ \delta_b(a, B, C) &= \sup\left\{\delta_b(a, b, C) : b \in B\right\}, \\ \delta_b(a, b, C) &= \inf\left\{d_b(a, b, c) : c \in C\right\}. \end{aligned}$$

Note that if for any $a \in A$, $b \in B$, $\delta_b(a, b, C) = 0$, then $a \in C$, $b \in C$.

Definition 3.7. Let (Y, d_b) be a tripled b-metric space. For any A, B and $C \in CL^b(Y)$, define tripled Hausdorff b-metric,

$$H_b(A,B,C) = \max\{\delta_b(A,B,C), \delta_b(A,C,B), \delta_b(B,A,C), \delta_b(B,C,A), \delta_b(C,A,B), \delta_b(C,B,A)\}.$$

Remark 3.8. For any $A, B, C, D \in CL^b(Y)$, we have

(i) $\delta_b(A,B,C) = 0$, then $A \subseteq C$, $B \subseteq C$;
(ii) $\delta_b(A,B,C) \leq s\,[\delta_b(A,B,D) + \delta_b(D,B,C)]$;
(iii) $H_b(A,B,C) = H_b(A,C,B) = H_b(B,A,C) = H_b(B,C,A) = H_b(C,A,B) = H_b(C,B,A)$;
(iv) $H(A,B,C) = 0$ then $A = B = C$;
(v) $H_b(A,B,C) \leq s\,[H_b(A,B,D) + H_b(D,B,C)]$.

Definition 3.9. Let (Y, d_b) be a tripled b-metric space and $\beta : Y \times Y \times Y \to [0,\infty)$ be a function. A mapping $S : Y \to CL^b(Y)$ is called tripled Hardy-Rogers-type (F, β)-contraction, if there exist $F \in \mathcal{F}$, and $r > 0$, such that

$$r + F\left(\beta(x,y,z)H(Sx,Sy,Sz)\right) \leq F\left(R(x,y,z)\right) \tag{6}$$

whenever $\min\{\beta(x,y,z)H(Sx,Sy,Sz), R(x,y,z)\} > 0$, where

$$R(x,y,z) = a_1 d_b(x,y,z) + a_2 d(x,Sy,z) + a_3 d_b(y,Sz,x) + a_4 d_b(z,Sx,y),$$

$a_1, a_2, a_3, a_4 \geq 0$, satisfying $a_1 + a_2 + (1+s)a_3 + s a_4 < s$.

Theorem 3.10. *Let (Y, d_b) be a complete tripled b-metric space with $s > 1$ and let $S : Y \to CL^b(Y)$ is a tripled Hardy-Rogers-type (F, β)-contraction such that the following conditions hold*

(i) *S is an β_s-admissible mapping;*
(ii) *There exist $y^0 \in Y$, $y_1 \in Sy^0$, $y_2 \in Sy_1$ with $\beta(y^0, y_1, y_2) \geq s^2$;*
(iii) *For any sequence $\{y_n\}$ in Y, such that $y_n \to x$ as $n \to \infty$, $\beta(y_n, y_{n+1}, y_{n+1}) \geq s^2$, for any $n \in \mathbb{N}$, we get $\beta(y_n, x, x) \geq s^2$ for each $n \in \mathbb{N}$.*

Then S has a fixed point.

Proof. By condition (ii), there exist $y^0 \in Y$, $y_1 \in Sy^0$ and $y_2 \in Sy_1$ with $\beta(x_0, y_1, y_2) \geq s^2$. If $y_1 \in Ty_1$, then y_1 is a fixed point of T. Let $y_1 \notin Sy_1$. As $\beta(x_0, y_1, y_2) \geq s^2$, there exist $x_3 \in Sy_2$, such that

$$s d_b(y_1, y_2, y_3) \leq \beta(x_0, y_1, y_2)\, H(Sx_0, Sy_1, Sx_2). \tag{7}$$

Since F strictly increasing, we get

$$F\left(sd_b\left(y_1,y_2,y_3\right)\right)\leq F\left(\beta\left(x_0,y_1,y_2\right)H\left(Tx_0,Sy_1,Sx_2\right)\right).$$

From (6), we have

$$\begin{aligned} r+F\left(sd_b\left(y_1,y_2,y_3\right)\right) &\leq r+F\big(\beta\left(x_0,y_1,y_2\right)H\left(Tx_0,Sy_1,Sx_2\right)\big)\\ &\leq F\left(a_1d_b\left(x_0,y_1,y_2\right)+a_2d_b\left(x_0,Sy_1,y_2\right)\right.\\ &\qquad a_3d_b\left(y_1,y_3,x_0\right)+a_4d_b\left(x_2,Sx_0,y_1\right))\\ &\leq F\left(a_1d_b\left(x_0,y_1,y_2\right)+a_2d_b\left(x_0,y_2,y_2\right)\right.\\ &\qquad a_3d_b\left(x_0,y_1,y_3\right)+a_4d_b\left(x_2,y_1,y_1\right)).\end{aligned}$$

From (T_4) we have $d_b\left(x_0,y_2,y_2\right),d_b\left(y_1,y_1,y_2\right)\leq d_b\left(x_0,y_1,y_2\right)$. By rectangular inequality, we have

$$\begin{aligned} &a_1d_b\left(x_0,y_1,y_2\right)+a_2d_b\left(x_0,y_2,y_2\right)\\ &\qquad\qquad+a_3d_b\left(x_0,y_1,y_3\right)+a_4d_b\left(y_1,y_1,y_2\right)\\ &\qquad\leq(a_1+a_2+a_3+sa_4)d_b\left(x_0,y_1,y_2\right)+sa_3d_b\left(y_1,y_2,y_3\right).\end{aligned}$$

Therefore, we have

(8)
$$r+F\left(sd_b\left(y_1,y_2,y_3\right)\right)\leq r+F\left((a_1+a_2+a_3+sa_4)d_b\left(x_0,y_1,y_2\right)+sa_3d_b\left(y_1,y_2,y_3\right)\right).$$

Since F is strictly increasing, we have from above that

$$\begin{aligned} sd_b\left(y_1,y_2,y_3\right)&\leq(a_1+a_2+a_3+sa_4)d_b\left(x_0,y_1,y_2\right)+sa_3d_b\left(y_1,y_2,y_3\right)\\ s(1-a_3)d_b\left(y_1,y_2,y_3\right)&\leq(a_1+a_2+a_3+sa_4)d_b\left(x_0,y_1,y_2\right)\\ d_b\left(y_1,y_2,y_3\right)&\leq\frac{a_1+a_2+a_3+sa_4}{s(1-a_3)}d_b\left(y_1,y_2,y_3\right).\end{aligned}$$

Take $\lambda=\frac{a_1+a_2+a_3+sa_4}{s(1-a_3)}$, we have $0\leq\lambda<1$ and $d_b\left(y_1,y_2,y_3\right)\leq\lambda d_b\left(x_0,y_1,y_2\right)<d_b\left(x_0,y_1,y_2\right)$. Now, from (8), we obtain

$$r+F\left(sd_b\left(y_1,y_2,y_3\right)\right)\leq F\left(d_b\left(x_0,y_1,y_2\right)\right).$$

Since S is β_s-admissible, we have $\beta\left(y_1,y_2,y_3\right)\geq s^3$. By this process way, we get a sequence $\{y_n\}\subset Y$ such that $y_n\in Sx_{n-1}$, $x_{n-1}\neq y_n$ and $\beta\left(x_{n-1},y_n,y_{n+1}\right)\geq s^2$. Furthermore,

$$r+F\left(sd_b\left(y_n,y_{n+1},y_{n+2}\right)\right)\leq F\left(d_b\left(x_{n-1},y_n,y_{n+1}\right)\right).$$

Thus by Lemma 3.8, $\{y_n\}$ is a Cauchy sequence in Y. As (Y,d_b) is complete, there exist $y^0\in Y$, such that $y_n\to y^0$ as $n\to\infty$. By condition (iii) we have

$\beta\left(y_n, y^0, y^0\right) \geq s^2$ fro each $n \in \mathbb{N}$ We claim that $d_b\left(y^0, y^0, Sy^0\right) = 0$. On contrary, suppose that $d_b\left(y^0, y^0, Sy^0\right) > 0$, there exist $n_0 \in \mathbb{N}$ such that $d_b\left(y_n, y_n, Sy^0\right) > 0$ for each $n \geq n_0$. For each $n \geq n_0$, we have

$$\begin{aligned} d_b\left(y^0, y_n, Sy^0\right) &\leq sd_b\left(y^0, y_n, y_{n+1}\right) + sd_b\left(y_{n+1}, y_n, Sy^0\right) \\ &\leq sd_b\left(y^0, y_n, y_{n+1}\right) + \beta\left(y_n, y_{n-1}, y^0\right) H\left(Sy_n, Sy_{n-1}, Sy^0\right) \\ &\leq sd_b\left(y^0, y_n, y_{n+1}\right) + a_1\left(y_n, y_{n-1}, y^0\right) + a_2\left(y_n, Sy_{n-1}, y^0\right) \\ &\quad + a_3\left(y_{n-1}, Sy^0, y_n\right) + a_4\left(y^0, Sy_n, y_{n-1}\right) \\ &\leq sd_b\left(y^0, y_n, y_{n+1}\right) + a_1 d_b\left(y_n, y_{n-1}, y^0\right) + a_2 d_b\left(y_n, y_n, y^0\right) \\ &\quad + a_3 d_b\left(x_{n-1}, Sy^0, y_n\right) + sa_4 d_b\left(y_n, y_{n+1}, y^0\right) + sa_4 d_b\left(y_n, y_{n+1}, y_{n-1}\right). \end{aligned} \tag{9}$$

Letting $n \to \infty$ in (9), we have

$$\begin{aligned} d_b\left(y^0, y^0, Sy^0\right) &\leq sd_b\left(y^0, y^0, y^0\right) + a_1 d_b\left(y^0, y^0, y^0\right) + a_2 d_b\left(y^0, y^0, y^0\right) \\ &\quad + a_3 d_b\left(y^0, y^0, Sy^0\right) + sa_4 d_b\left(y^0, y^0, y^0\right) + sa_4 d_b\left(y^0, y^0, y^0\right). \end{aligned}$$

Consequently we have $d_b\left(y^0, y^0, Sy^0\right) \leq a_3 d_b\left(y^0, y^0, Sy^0\right)$. It is a contradiction. Thus $d_b\left(y^0, y^0, Sy^0\right) = 0$ and $y^0 \in Sy^0$. □

Example 3.11. Let $Y = \mathbb{Z}$ be endowed with a tripled $b-$metric,

$$d(x, y, z) = \max\left\{|x-y|^2, |x-z|^2, |y-z|^2\right\}$$

for each $x, y, z \in Y$, with $s = 2$. Define $S : Y \to CL^b(Y)$,

$$Sx = \begin{cases} \{-x, -x+1\}, & x < 0, \\ \{0, 1\}, & x = 0, \\ \{-x, -x+1\}, & x > 0, \end{cases}$$

and $\beta : Y \times Y \times Y \to [0, \infty)$ as follows

$$\beta(x, y, z) = \begin{cases} 4, & x, y, z \in \{0, 1\}, \\ \frac{1}{5}, & x, y, z > 1, \\ 0, & otherwise. \end{cases}$$

Take $F(x) = \ln x$, for each $x \in (0, \infty)$. By this F, condition (6) in Definition 3.9 reduce to

$$\frac{\beta(x, y, z) H(Sx, Sy, Sz)}{R(x, y, z)} \leq e^{-r}.$$

For any $x, y, z \in Y$ with $\min\{\beta(x, y, z) H(Sx, Sy, Sz), R(x, y, z)\} > 0$. Assume that $a_1 = \frac{3}{2}$, $a_2 = a_3 = a_4 = 0$. We consider the following case. If $x = 2$, $y = 3$, $z = 4$, we have $\beta(x, y, z) = \frac{1}{5}$, $H_d(S2, S3, S4) = H_d(\{-2, -1\}, \{-3, -2\}, \{-4, -3\})$.

Case 1.

$$\begin{aligned}\delta_d(\{-2,-1\},\{-3,-2\},\{-4,-3\}) &= \max\{\delta_d(-2,\{-3,-2\},\{-4,-3\}),\\ &\qquad \delta_d(-1,\{-3,-2\},\{-4,-3\})\} = 4,\\ \delta_d(-2,\{-3,-2\},\{-4,-3\}) &= \max\{\delta_d(-2,-3,\{-4,-3\}),\\ &\qquad \delta_d(-2,-2,\{-4,-3\})\} = 1,\\ \delta_d(-1,\{-3,-2\},\{-4,-3\}) &= \max\{\delta_d(-1,-3,\{-4,-3\}),\\ &\qquad \delta_d(-1,-2,\{-4,-3\})\} = 4.\end{aligned}$$

Case 2.

$$\begin{aligned}\delta_d(\{-2,-1\},\{-4,-3\},\{-3,-2\}) &= \max\{\delta_d(-2,\{-4,-3\},\{-3,-2\}),\\ &\qquad \delta_d(-1,\{-4,-3\},\{-3,-2\})\} = 9,\\ \delta_d(-2,\{-4,-3\},\{-3,-2\}) &= \max\{\delta_d(-2,-4,\{-3,-2\}),\\ &\qquad \delta_d(-2,-3,\{-3,-2\})\} = 4,\\ \delta_d(-1,\{-4,-3\},\{-3,-2\}) &= \max\{\delta_d(-1,-4,\{-3,-2\}),\\ &\qquad \delta_d(-1,-3,\{-3,-2\})\} = 9.\end{aligned}$$

Case 3.

$$\begin{aligned}\delta_d(\{-3,-2\},\{-2,-1\},\{-4,-3\}) &= \max\{\delta_d(-3,\{-2,-1\},\{-4,-3\}),\\ &\qquad \delta_d(-2,\{-2,-1\},\{-4,-3\})\} = 4,\\ \delta_d(-3,\{-2,-1\},\{-4,-3\}) &= \max\{\delta_d(-3,-2,\{-4,-3\}),\\ &\qquad \delta_d(-3,-1,\{-4,-3\})\} = 4,\\ \delta_d(-2,\{-2,-1\},\{-4,-3\}) &= \max\{\delta_d(-2,-2,\{-4,-3\}),\\ &\qquad \delta_d(-2,-1,\{-4,-3\})\} = 4.\end{aligned}$$

Case 4.

$$\begin{aligned}\delta_d(\{-3,-2\},\{-4,-3\},\{-2,-1\}) &= \max\{\delta_d(-3,\{-4,-3\},\{-2,-1\}),\\ &\qquad \delta_d(-2,\{-4,-3\},\{-2,-1\})\} = 4,\\ \delta_d(-3,\{-4,-3\},\{-2,-1\}) &= \max\{\delta_d(-3,-4,\{-2,-1\}),\\ &\qquad \delta_d(-3,-3,\{-2,-1\})\} = 4,\\ \delta_d(-2,\{-4,-3\},\{-2,-1\}) &= \max\{\delta_d(-2,-4,\{-2,-1\}),\\ &\qquad \delta_d(-2,-3,\{-2,-1\})\} = 4.\end{aligned}$$

Case 5.

$$\begin{aligned}\delta_d(\{-4,-3\},\{-3,-2\},\{-2,-1\}) &= \max\{\delta_d(-4,\{-3,-2\},\{-2,-1\}),\\ &\qquad \delta_d(-3,\{-3,-2\},\{-2,-1\})\} = 4,\\ \delta_d(-4,\{-3,-2\},\{-2,-1\}) &= \max\{\delta_d(-4,-3,\{-2,-1\}),\\ &\qquad \delta_d(-4,-2,\{-2,-1\})\} = 4,\\ \delta_d(-3,\{-3,-2\},\{-2,-1\}) &= \max\{\delta_d(-3,-3,\{-2,-1\}),\\ &\qquad \delta_d(-3,-2,\{-2,-1\})\} = 1.\end{aligned}$$

Case 6.

$$\begin{aligned}\delta_d(\{-4,-3\},\{-2,-1\},\{-3,-2\}) &= \max\{\delta_d(-4,\{-2,-1\},\{-3,-2\}),\\ &\qquad \delta_d(-3,\{-2,-1\},\{-3,-2\})\} = 4,\\ \delta_d(-4,\{-2,-1\},\{-3,-2\}) &= \max\{\delta_d(-4,-2,\{-3,-2\}),\\ &\qquad \delta_d(-4,-1,\{-3,-2\})\} = 4,\\ \delta_d(-3,\{-2,-1\},\{-3,-2\}) &= \max\{\delta_d(-3,-2,\{-3,-2\}),\\ &\qquad \delta_d(-3,-1,\{-3,-2\})\} = 1.\end{aligned}$$

We conclude that $H_d(\{2,3\},\{3,4\},\{4,5\}) = 9$,

$$\frac{\beta(2,3,4)H(S2,S3,S4)}{R(2,3,4)} \leq e^{-r},$$

so, $e^r \leq \frac{10}{3}$, or $r \leq \ln\frac{10}{3}$. Therefore, $R(2,3,4) = \frac{3}{2}d_b(2,3,4) = 4 \times \frac{3}{2} = 6$. Thus S is tripled Hardy-Rogres-type (F,β)-contraction with $F(x) = \ln x$, $y^0 = 1$, we have $y_1 \in \{0,1\}$, if $y_1 = 0$ then $y_2 \in S0 = \{0,1\}$, $\beta(y^0,x)1,y_2) = 4$. This is obvious to show that S is β_s-admissible mapping and for any $\{y_n\} \subseteq Y$ such that $y_n \to x$ as $n \to \infty$, and $\beta(y_n,y_{n+1},y_{n+2}) = 4$ for any $n \in \mathbb{N}$. However we have $\beta(y_n,x,x) = 4$ for each $n \in \mathbb{N}$. Therefore, all conditions of Theorem 3.10 are satisfied and S has a fixed point in Y.

Definition 3.12. Let (Y,d_b) be a tripled b-metric space and $\beta : Y \times Y \times Y \to [0,\infty)$ be a function. A mapping $S : Y \to CL^b(Y)$ called tripled Hardy-Rogers-type (F,β)-contraction, if there exist $F \in F_s$, and $r > 0$ such that

$$r + F\left(\alpha^\star(Sx,Sy,Sz)H(Sx,Sy,Sz)\right) \leq F\left(R(x,y,z)\right)$$

for each $x, y, z \in Y$, whenever $\min\{\alpha^\star(Sx, Sy, Sz)H(Sx, Sy, Sz), R(x, y, z)\} > 0.$, where

$$R(x, y, z) = a_1 d_b(x, y, z) + a_2 d_b(x, Sy, z) + a_3 d_b(y, Sz, x) + a_4 d_b(z, Sx, y)$$

with $a_1, a_2, a_3, a_4 \geq 0$ satisfiying $a_1 + a_2 + (1 + s)a_3 + sa_4 < s$.

Theorem 3.13. *Let (Y, d_b) be a complete tripled b-metric space with $s > 1$, and let $S : Y \to CL^b(Y)$ be $(F, \beta^\star)$-contraction of tripled Hardy-Rogres-type such that the following conditions hold*

- (i) *S is an $\beta_s^\star$-admissible mapping;*
- (ii) *There exist $y^0 \in Y$ and $y_1 \in Sy^0$ and $y_2 \in Sy_1$ with $\beta(y^0, y_1, y_2) \geq s^2$;*
- (iii) *For any sequence $\{y_n\}$ in Y such that $y_n \to x$, as $n \to \infty$, and $\beta(y_n, y_{n+1}, y_{n+2}) \geq s^2$ for any $n \in \mathbb{N}$, we have $\beta(y_n, x, x) \geq s^2$ for any $n \in \mathbb{N}$.*

Then S has a fixed point.

Proof. The proof of this theorem tuns along the same lines as the proof of Theorem 3.10. □

4. Application

In this section, we obtain existence theorems for Voltrra-type. For this purpose, let $Y = C([0, 1], \mathbb{R})$ be the space of all continuous real-valued functions on $[0, 1]$. Notice that Y is complete tripled b-metric space by considering

$$d_b(x, y, z) = \max\left\{\sup_{t\in[0,1]} |x(t) - y(t)|^2, \sup_{t\in[0,1]} |y(t) - z(t)|^2, \sup_{t\in[0,1]} |x(t) - z(t)|^2\right\},$$

for $s = 2$. Suppose the Volterra-type integral inclusion as

$$\begin{aligned} x(t) &= \int_0^t N(t, s, x(s))\, ds + g(t), \\ y(t) &= \int_0^t N(t, s, y(s))\, ds + g(t), \qquad (10) \\ z(t) &= \int_0^t N(t, s, z(s))\, ds + g(t), \end{aligned}$$

for $t \in [0, 1]$, where $N : [0, 1] \times [0, 1] \times \mathbb{R} \to A_{CV}$, and A_{CV} denotes the family of nonempty compact and convex subsets of $\mathbb{R}$. For each $x \in C([0, 1], \mathbb{R})$, the operator $N(., ., x)$ is lower semi-continuous. Further the function $g : [0, 1] \to \mathbb{R}$ is continuous. For the integral inclusion given above, we can define a multi-valued operator $S : C([0,1],\mathbb{R}) \to CL^b\left(C([0,1],\mathbb{R})\right)$ as follows

$$Sx(t) = \left\{ u \in C([0,1],\mathbb{R}) : u \in \int_0^t N(t,s,x(s))\,ds + g(t), t \in [0,1] \right\}.$$

Let $x \in C([0,1],\mathbb{R})$ and denote $N_x = N(t,s,x(s))$, $t,s \in [0,1]$. Now, for $N_x : [0,1] \times [0,1] \to A_{CV}$, by Michael's selection theorem, there exist a continuous operator $n_x : [0,1] \times [0,1] \to \mathbb{R}$ with $n_x(t,s) \in N_x(t,s)$ for each $t,s \in [0,1]$. This explain that $\int_0^t n_x(t,s)\,ds + g(t) \in Sx(t)$. The operator Sx is nonempty. Clearly, the operator Sx is closed, by [35].

Theorem 4.1. *Let $Y = C([0,1],\mathbb{R})$ and let the multi-valued operator $S : Y \to CL^b(Y)$,*

$$Sx(t) = \left\{ u \in Y : u \in \int_0^t N(t,s,x(s))\,ds + g(t), t \in [0,1] \right\}$$

where the function $g : [0,1] \to \mathbb{R}$ is contiunous and $N : [0,1] \times [0,1] \times \mathbb{R} \to A_{CV}$ is such that for each $x \in C([0,1],\mathbb{R})$, the operator $N(.,.,x)$ is lower semi-continuous. Assume that the following conditions satisfying.

(i) *There exist a continuous mapping $q : Y \to [0,\infty)$ such that*

$$H\left(N(t,s,x(s)), N(t,s,y(s)),\ N(t,s,z(s))\right) \le q(s) \max\left\{ \sup_{s\in[0,1]} |x(s)-y(s)|, \sup_{s\in[0,1]} |x(s)-z(s)|, \sup_{s\in[0,1]} |y(s)-z(s)| \right\}, \tag{11}$$

$$H\left(N(t,s,x(s)), N(t,s,y(s))\right) \le q(s) \sup_{s\in[0,1]} |x(s)-y(s)|, \tag{12}$$

$$H\left(N(t,s,y(s)), N(t,s,z(s))\right) \le q(s) \sup_{s\in[0,1]} |y(s)-z(s)|, \tag{13}$$

$$H\left(N(t,s,x(s)), N(t,s,z(s))\right) \le q(s) \sup_{s\in[0,1]} |x(s)-z(s)|, \tag{14}$$

for each $t,s \in [0,1]$, $x,y,z \in Y$;

(ii) *There exist $r > 0$ and $\beta : Y \times Y \times Y \to [0,\infty)$, such that for each $x,y,z \in Y$, we get*

$$\int_0^t q(s)\,ds \le \sqrt{\frac{e^{-r}}{\beta(x,y,z)}} \tag{15}$$

for $t \in [0,1]$;

(iii) *There exist $y_0 \in Y$, $y_1 \in Sy_0$ and $y_2 \in Sy_1$ with $\beta(y_0,y_1,y_2) \ge 4$;*

(iv) *If* $x \in Y$, $y \in Sx$, $z \in Sy$ *and* $w \in Sz$, *and so* $\beta(x,y,z) \geq 4$, *then we have* $\beta(y,z,w) \geq 4$*;*

(v) *For any sequence* $\{y_n\} \subset Y$ *such that* $y_n \to x$ *as* $n \to \infty$ *and* $\beta\left(y_n, y_{n+1}, y_{n+2}\right) \geq 4$ *for any* $n \in \mathbb{N}$, *we get* $\beta\left(y_n, x, x\right) \geq 4$ *for any* $n \in \mathbb{N}$.

Then, the integral inclusion has a solution.

Proof. We have to show that the operator S satisfies all conditions of Theorem 3.10. First, we check (i). Let $x, y, z \in Y$ be such that $u \in Sx$. Then we have $n_x(t,s) \in M_x(t,s)$ for $t, s \in [0,1]$ such that

$$u(t) = \int_0^t n_x(t,s)\,ds + g(t),$$

for $t \in [0,1]$. On the other hand, hypothesis (i) that, there exist $v(t,s) \in N_y(t,s)$ and $w(t,s) \in N_z(t,s)$, such that

$$\begin{aligned} |n_x(t,s) - v(t,s)| &\leq q(s)|x(s) - y(s)|, \\ |n_z(t,s) - v(t,s)| &\leq q(s)|z(s) - y(s)|, \end{aligned}$$

and

$$|n_x(t,s) - w(t,s)| \leq q(s)|x(s) - z(s)|.$$

Let us consider the multi-valued operator U defined by

$$U(t,s) = N_y(t,s) \cap \{\lambda \in \mathbb{R} : |n_x(t,s) - \lambda| \leq q(s)|x(s) - y(s)|, t, s \in [0,1]\}.$$

Since the operator S is lower semi-continuous, there exist $n_y : [0,1] \times [0,1] \to \mathbb{R}$ such that $n_y(t,s) \in U(t,s)$ for each $t, s \in [0,1]$. Then we get

$$u_1(t) = \int_0^t n_y(t,s)\,dt + g(t) \in \int_0^t N(t,s,y(s))\,ds + g(t),$$

and for each $t \in [0,1]$, we have

$$\begin{aligned} |u(t) - u_1(t)|^2 &\leq \left(\int_0^t |n_x(t,s) - n_y(t,s)|\,ds\right)^2 \\ &\leq \left(\int_0^t q(s)|x(s) - y(s)|\,ds\right)^2 \\ &\leq \left(\sqrt{\sup_{s \in [0,1]} |s(x) - y(s)|} \int_0^t q(s)\,ds\right)^2. \end{aligned} \tag{16}$$

In this way we get

$$u_2(t) = \int_0^t n_z(t,s)\,dt + g(t) \in \int_0^t M(t,s,z(s))\,ds + g(t),$$

such that

$$|u_1(t) - u_2(t)|^2 \leq \left(\sqrt{\sup_{s\in[0,1]} |y(s) - z(s)|} \int_0^t q(s)\, ds \right)^2, \tag{17}$$

and

$$|u(t) - u_2(t)|^2 \leq \left(\sqrt{\sup_{s\in[0,1]} |x(s) - z(s)|} \int_0^t q(s)\, ds \right)^2. \tag{18}$$

Now, From (16), (17) and (18), we have

$$\begin{aligned} \max\left\{|u(t) - u_1(t)|^2, |u_1(t) - u_2(t)|^2,\ |u(t) - u_2(t)|^2\right\} &\leq \max\Bigg\{ \sup_{s\in[0,1]} |x(s) - y(s)|^2, \\ &\qquad \sup_{s\in[0,1]} |y(s) - z(s)|^2, \sup_{s\in[0,1]} |x(s) - z(s)|^2 \Bigg\} \\ &\leq d_b(x, y, z) \left(\int_0^t q(s)\, ds \right)^2 \\ &\leq d_b(x, y, z) \frac{e^{-r}}{\beta(x, y, z)}. \end{aligned}$$

Consequently, we have

$$\beta(x, y, z) d_b(u, u_1, u_2) \leq d_b(x, y, z) e^{-r}.$$

At present, by just interchanging the role x, y and z, we reach

$$\beta(x, y, z) H_b(Sx, Sy, Sz) \leq e^{-r} d_b(x, y, z).$$

As natural logarithm belongs to S and applying it on above inequality and some simplification, we get

$$r + \ln\left(\beta(x, y, z) H(Sq, Sg, Sz)\right) \leq \ln\left(\beta(x, y, z)\right).$$

Thus, S is (F, β)-contraction of tripled Hardy-Rogers-type with $a_1 = 1$, $a_2 = a_3 = a_4 = 0$ and $F(x) = \ln x$. All other conditions of Theorem 3.10 immediately follows by the hypothesis. Therefore, the operator S has a fixed dpoint. □

Conclusion

We defined a tripled b-metric space and tripled Hausdorff b-metric space and give many examples about them. Finally, we apply our results to get existence theorems for Volterra-type integral inclusion in tripled Hausdorff m-metric spaces.

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