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BEST PROXIMITY POINT RESULTS VIA GENERALIZED CONTRACTIONS

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Abstract. This paper extends best proximity point theorems in metric spaces by incorporating the notion of generalized ψ-ϕ contractions. We establish results for a single and a pairs of non-self mappings and demonstrating that the existence and uniqueness of best proximity points under some specific conditions. The obtained results also provide convergence results for sequences generated by these mappings. Additionally, we discuss how these results are related to the celebrated classical Banach contraction principle in the settings of metric spaces.

1. Introduction

In the study of the metrical fixed point, the notable Banach contraction principle (Bcp) [4] demonstrated that the contraction mapping has exactly one fixed point on the complete metric spaces. This well-known principle has seen various extensions and generalizations on the metric spaces and other abstract settings. The study of invariance of the point whichever transformation it undergoes has been a cornerstone in nonlinear analysis. Evolving further, notion of best proximity point came into existence to address the cases of non-existence of fixed point. The study of best proximity points ensures optimal solutions when fixed points are absent.

In the study of set-valued mappings, K. Y. Fan [7] presented an extension of Bcp, providing better understanding how fixed points behave beyond simple contractions in complex settings. Also, Boyd and Wong [6] extended the Bcp for non-linear contraction in the settings of metric spaces which is further utilized by several authors, we refer some new interesting generalizations in this direction ([2, 8, 9, 10]). This

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study transformed into the generalization of the concept of various non-linear mappings, which cover the various applicability of such findings in context of non-linear operators and optimization problems, see for instance ([11, 12, 14, 15, 17]). These extensions were further unified by Vetrivel [18] and the process of these generalizations are still on.

Meanwhile, significant contributions by Eldred and Veermani [3] emerged with establishing results for cyclic contractions within the framework of uniformly convex Banach spaces. These results gained attention due to their more general form and the relaxed continuity conditions involved. It has shown that for a non-self mapping between non-empty subsets of a metric space there exist a point called best proximity point, having distance with its image is minimum. Basha [13] further extended this concept, by introducing non-self proximal contraction and proving best proximity point results. This concept highlighted the importance of understanding the behavior of mappings between subsets of metric spaces, paving the way for deeper explorations of contractions in complex environments.

Sanhan [16] introduced the concept of generalized ψ-contraction allowing a more flexible contraction condition. It encompasses a wider class of mappings, enhancing the applicability and impact of the fixed point theory. Recently, Aftab Alam [1] introduced notions of T-proximal sequence, proximally completeness, proximally closedness and proximally continuity besides generalizing the results obtained by Basha. The requirement for the entire metric space to be complete has been replaced by the proximal completeness of the subspace A. Additionally, the condition that the subsets A and B of metric space must be closed has been relaxed by proximal closedness for the same. The proximal closedness of A_0, a relatively weaker notion, has been used in place of the approximative compactness of B with respect to A. Moreover, the proximal continuity of T further relaxes the necessity of this condition too.

In this way, we prove best proximity point results for generalized $\psi-\phi$ contraction using relatively weaker metrical notions introduced by Aftab Alam [1]. This extension unifies and broadens the scope of existing theories by providing best proximity point results, further enriching the understanding of non-self mappings in metric spaces as well.

2. Preliminaries

Throughout this paper, we will use the following notations, for a given pair (A, B) of nonempty subsets of a metric space :

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$
$$d(x, B) = \inf\{d(x, y) : y \in B\}, \quad \text{for } x \in A,$$
$$A_0 = \{x \in A : d(x, y) = d(A, B), \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B), \text{ for some } x \in A\}.$$

Notably, for each $x \in A_0$, $\exists y \in B_0$ such that $d(x, y) = d(A, B)$ and conversely, for each $y \in B_0$, $\exists x \in A_0$ such that $d(x, y) = d(A, B)$. Hence, A_0 is nonempty if and only if B_0 is nonempty. Also, whenever A and B intersects, A_0 and B_0 being nonempty.

Definition 2.1. [3] Let (X, d) be a metric space and (A, B) a pair of two nonempty subsets of X. An element $x \in A$ is called a *best proximity point* for the mapping $T : A \to B$ if it satisfies

$$d(x, Tx) = d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

In other words, $x \in A$ is a best proximity point of T if the function $d(x, Tx)$ achieves its global minimum at x with the value $d(A, B)$.

Definition 2.2. [5] Let (X, d) be a metric space and (A, B) a pair of nonempty subsets of X. A mapping $T : A \to B$ is called a *proximal contraction* if $\exists\ \epsilon \in [0, 1)$ such that for all $x, y, u, v \in A$,

$$d(u, Tx) = d(v, Ty) = d(A, B) \Rightarrow d(u, v) \leq \epsilon\ d(x, y).$$

Remark 2.3. It is not necessary for proximal contraction to be continuous. However, when restricted to the case where $A = B = X$, a proximal contraction is equivalent to a standard contraction.

Theorem 2.4. [5] *Let (X, d) be a complete metric space, (A, B) a pair of non-empty subsets of X, and $T : A \to B$ a mapping. Suppose that the following conditions hold:*

(1) $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$,
(2) $T(A_0) \subseteq B_0$,
(3) T is a proximal contraction,
(4) A and B are closed subspaces of X,

(5) B is approximatively compact with respect to A.

Then T has a unique best proximity point. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B)$$

Definition 2.5. Let (X, d) be a metric space, and let (A, B) be a pair of nonempty subsets of X. Consider a mapping $T : A \to B$. A sequence $\{x_n\} \subset A$ is referred to as *T-proximal* if it satisfies the condition

$$d(x_{n+1}, Tx_n) = d(A, B).$$

Remark 2.6. For $A = B = X$, T-proximal sequence is a sequence of Picard iterations.

Definition 2.7. Let (X, d) be a metric space, and (A, B) a pair of nonempty subsets of X. Consider a mapping $T : A \to B$. The subspace (A, d) is said to be *proximally complete* if every T-proximal Cauchy sequence in A converges within A. It is evident that every complete subspace of a metric space is also proximally complete.

Definition 2.8. Let (X, d) be a metric space, with (A, B) as a pair of nonempty subsets of X, and $T : A \to B$ as a mapping. For a subset $E \subseteq A$, we say that E is a *proximally closed subspace* of A if the limit of every T-proximal convergent sequence in E lies within E. It is clear that every closed subspace of a metric space is also proximally closed.

Definition 2.9. Let (X, d) be a metric space and (A, B) a pair of nonempty subsets of X. A mapping $T : A \to B$ is called *proximally continuous* at a point $x \in A$ if for any T-proximal sequence $\{x_n\} \subset A$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. T is called *proximally continuous* if it is proximally continuous at each point of X.

Definition 2.10. [13] A pair of mappings (N, H), where $N : A \to B$ and $H : B \to A$, is said to be a *proximal cyclic contraction pair* if there exists a non-negative number $\alpha < 1$ such that for all $u, x \in A$ and $v, y \in B$,

$$d(Nu, v) + d(Hv, x) \leq \alpha[d(u, y) + d(x, v)].$$

3. MAIN RESULTS

In this section, we consider main results, examples and future direction of this research work.

$\Psi : \{\psi : [0,\infty) \to [0,\infty);$ such that ψ in non decreasing continuous function$\}$
$\Phi : \{\phi : [0,\infty) \to [0,\infty);$ such that ϕ is lower semicontinuous$\}$
where $\psi(t) = \phi(t) = 0$ if and only if $t = 0$.

Definition 3.1. Let A and B be two non-empty subsets of a metric space (X, d). A non-self mapping $N : A \to B$ is said to be a *generalized proximal ψ-ϕ-contraction* if for all $x, y, u, v \in P$, the following condition is satisfied:

$$d(u, Nx) = d(v, Ny) = d(P, Q) \implies \psi(d(u, v)) \le \psi(d(x, y)) - \phi(d(x, y)).$$

Theorem 3.2. *Let (X, d) be a metric space (A, B) a pair of non-empty subsets of X and $T : A \to B$ a non-self mapping. Suppose that the following conditions hold:*
(a) A_0 is non empty,
(b) $T(A_0) \subseteq B_0$,
(c) T is a generalized $\psi - \phi$ contraction,
(d) A is proximally complete subspace of X,
(e) either T is proximally continuous or A_0 is proximally closed in A,
then T has a unique best proximity point. furthermore, for any fixed $x_0 \in A_0$, the $T-$proximal sequence $\{x_n\}$ based on x_0 converges to the unique best proximity point of T.

Proof. Since A_0 is non empty, there exist $x_0 \in A_0$ and by definition $T(x_0) \in B_0$, therefore, there exists $x_1 \in A_0$ such that $d(x_0, T(A_0)) = d(A, B)$.
By continuing this process construct a sequence $\{x_n\} \subset A_0$, such that

$$d(x_{n+1}, T(x_n)) = d(A, B).$$

Now, applying generalized $\psi - \phi$ contraction condition for all $n \in \mathbb{N}_0$, we get

$$\psi(d(x_{n+1}, x_{n+2})) \le \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1}))$$

Let $a_n = d(x_n, x_{n+1})$ then $\psi(a_{n+1}) \le \psi(a_n) - \phi(a_n)$, Since ψ is non decreasing and ϕ is lower semi-continuous, this implies $\{\psi(a_n)\}$ is non-increasing sequence.
Since ψ is non-decreasing and $\{\psi(a_n)\}$ is non-increasing and bounded below, it converges thus,

$$\lim_{n\to\infty} \psi(a_n) = l$$

where l is non negative real number. If $l > 0$, then there exist $\epsilon > 0$ such that $\phi(a_n) > \epsilon$ for all n which is contradicting convergence of $\psi(a_n)$, hence

$$l = 0 \text{ and } \lim_{n\to\infty} \psi(a_n) = 0.$$

Since ψ being continuous $lim_{n\to\infty}a_n = 0$ so, we can write $\lim_{n\to\infty} \psi(a_n) = 0$.
Now, to show that $\{x_n\}$ is cuchy sequence, we use triangle inequality for $m > n$,

$$d(x_m, x_n) \ \leq \ d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n),$$

Since $d(x_{k+1}, x_k)$ tends to 0 as k tends to ∞ for some $\epsilon > 0$ there exists $N > 0$ such that for $k \geq N$, $d(x_{k+1}, x_k) < \frac{\epsilon}{m-n}$. For $m > n$

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) < \sum_{k=n}^{m-1} \frac{\epsilon}{m-n} = \epsilon.$$

which implies $\{x_n\}$ is cauchy sequence.
Now A being proximally complete there exist $x \in A$ such that $x_n \to x$.
Case I: T is proximally continuous.
Since $\{x_n\}$ is a T-proximal sequence satisfying $x_n \to x$ the proximal continuity of T implies $T(x_n) \to T(x)$.
Now, using the continuity of d and definition of T-proximal sequence, we get,

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, T(x_n)) = d(A, B),$$

hence, x is best proximity point of T.
Case II : Since $\{x_n\} \subset A_0$ is a $T-$proximal sequence satisfying $x_n \to x \in A$. We have, $x \in A_0$ and by assumption $T(x_0) \in B_0$. Therefore, there exist $\omega \in A_0$ such that $d(x_{n+1}, \omega) \to 0$ as $n \to \infty$. So, $x_n \to \omega$ and by uniqueness of limit we can write $\omega = x$. Hence, x is best proximity point of T.
Suppose x and y are two best proximity points then $d(x, T(x)) = d(y, T(y)) = d(A, B)$. Applying $\psi - \phi$ contraction we get,

$$\psi(d(x, y)) \leq \psi(d(x, y)) - \phi(d(x, y)),$$

which implies $\phi(d(x, y)) = 0$. Since ϕ is lower semi-continuous and $\phi(r) = 0$ if and only if $r = 0$, we get, $d(x, y) = 0$ resulting in $x = y$. Hence, x is the unique best proximity point. □

Remark 3.3. Under the restriction A = B = X, our theorem reduces to classical Banach contraction principle.

Example 3.4. Let (X, d) be a metric space, where $X = \mathbb{R}$ and $d = |x - y|$. Define a non self map $T : A \to B$ by $T(x) = 4 - \frac{x}{2}$, $A = [0, 2], B = [3, 5]$, here $d(A, B) = 1$. For $x = 0 \in A$, $T(0) = 4 \in B$ and, for $x = 2 \in A$, $T(2) = 4 - \frac{2}{2} = 4 - 1 = 3 \in B$.

Now, $d(Tx,Ty) = |4 - \frac{x}{2} - (4 - \frac{y}{2})| = |\frac{y-x}{2}| = \frac{1}{2}d(x,y)$, also $\psi(\frac{d(x,y)}{2}) = \frac{d(x,y)}{2}$.
Now,

$$\frac{d(x,y)}{2} \leq d(x,y) - \frac{d(x,y)}{2},$$
$$\frac{d(x,y)}{2} \leq \frac{d(x,y)}{2}.$$

Now, finding a point x^* such that $d(x^*, T(x^*)) = d(A,B)$, by simplifying

$$d(x^*, 4 - \tfrac{x^*}{2}) = 1,$$
$$|x^* - (4 - \tfrac{x^*}{2})| = 1$$
$$|\frac{3x^*}{2} - 4| = 1$$
$$x^* = \frac{10}{3} \notin B \text{ or } x^* = 2 \in B$$

Hence $x^* = 2$ is the unique best proximity point.

Theorem 3.5. *Let (X,d) be a metric space, (A,B) a pair of non-empty subsets of X and $N : A \to B$, $H : B \to A$ be non-self mappings. Suppose the following conditions hold,*
(a) A_0 and B_0 are non-empty,
(b) $N(A_0) \subseteq B_0$ and $H(B_0) \subseteq A_0$,
(c) N and H are generalized $\psi - \phi$ contractions,
(d) the pair (N,H) is proximal cyclic contraction,
(e) A is proximally complete,
(f) either N is proximally continuous or A_0 is proximally closed in A,
then there exist unique points $x \in A$ and $y \in B$ such that $d(Nx,x) = d(Hy,y) = d(A,B)$. Furthermore, for any fixed $x_0 \in A_0$ and $y_0 \in B_0$, the $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = Nx_n$ and $y_{n+1} = Hy_n$, converges to x and y respectively.

Proof. We need to prove the existence and uniqueness of points $x \in A$ and $y \in B$ such that

$$d(Nx,x) = d(Hy,y) = d(A,B)$$

and the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = Nx_n$ and $y_{n+1} = Hy_n$ converges to x and y respectively.
Constructing sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$x_{n+1} = Nx_n,\ y_{n+1} = Hy_n,$$

Since $N(A_0) \subseteq B_0$ and $H(B_0) \subseteq A_0$, the sequences $\{x_n\}$ and $\{y_n\}$ remain 0 in A_0 and B_0 respectively.
Now, we shall show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. For this, we utilize the fact that N and H are generalized $\psi - \phi$ contractions, we have

$$\psi d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1})) - \phi(d(x_n, x_{n-1})).$$

Since ψ is non-decreasing the sequence $\{d(x_{n,x_{n-1}})\}$ is non-increasing and converges to some $t \geq 0$.
Assume that $t > 0$, now taking the limit as $n \to \infty$, we get

$$\psi(t) \leq \psi(t) - \phi(t),$$

this implies $\phi(t) = 0$ which contradicts definition of ϕ, hence $t = 0$. Thus,

$$\lim_{n \to \infty} d(x_n, x_{n-1}) = 0.$$

Similarly,

$$\lim_{n \to \infty} d(y_n, y_{n-1}) = 0.$$

Since A is proximally complete $\{x_n\}$ converges to some $x \in A$ and $\{y_n\}$ converges to some $y \in B$.
We claim that, $d(Nx, x) = d(Hy, y) = d(A, B)$, since, $\{x_n\}$ and $\{y_n\}$ converges to x and y respectively, so by using continuity of d we can write,

$$\lim_{n \to \infty} d(Nx_n, x_{n+1}) = d(Nx, x),$$
$$\lim_{n \to \infty} d(Hy_n, y_{n+1}) = d(Hy, y).$$

Given that $d(Nx_n, x_{n+1}) = d(A, B)$ for all $n \in \mathbb{N}$, we have

$$d(Nx_n, x_{n+1}) = d(A, B),$$
$$d(Hy_n, y_{n+1}) = d(A, B).$$

Now, to prove the uniqueness of the points x and y assume $x' \in A$ and $y' \in B$ are also points such that, $d(Nx', x') = d(Hy', y') = d(A, B)$, using the proximal cyclic contraction property we get,

$$\psi(d(x', x)) \leq \psi(d(x', x)) - \phi(d(x', x)),$$

Since ϕ is positive we get $d(x', x) = 0$ which implies $x' = x$. Similarly we can prove $y' = y$. Thus x, y are unique. □

Example 3.6. Let (X, d) be a metric space, $X = \mathbb{R}^2$ and $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Let $A = \{(x, 0) : 0 \leq x \leq 1\}$ and $B = \{(x, 1) : 0 \leq x \leq 1\}$ be disjoint sets in $\mathbb{R}^2$. Define mappings $N : A \to B$ and $H : B \to A$ as follows,

$$N((x,0)) = (x,1) \text{ for all } (x,0) \in A,$$
$$H((x,1)) = (x,0) \text{ for all } (x,1) \in B.$$

Let $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$.
Now we have,

$$A_0 = \{(0,0),(1,0)\} \text{ and } B_0 = \{(0,1),(1,1)\},$$
$$N(A_0) = \{(0,1),(1,1)\} \text{ and } N(B_0) = \{(0,0),(1,0)\}.$$

Let $A_0 = \{(0,0),(1,0)\}$ and $B_0 = \{(0,1),(1,1)\}$. A and B are non-empty.

- For N: For any $(x_1, 0), (x_2, 0) \in A$, we have $d(N((x_1, 0)), N((x_2, 0))) = d((x_1, 1), (x_2, 1)) = |x_1 - x_2|$. Thus,

$$\psi(d(N((x_1,0)), N((x_2,0)))) = |x_1 - x_2| \leq |x_1 - x_2| - \frac{|x_1 - x_2|}{2} = \frac{|x_1 - x_2|}{2}.$$

- For H: For any $(x_1, 1), (x_2, 1) \in B$, we have $d(H((x_1, 1)), H((x_2, 1))) = d((x_1, 0), (x_2, 0)) = |x_1 - x_2|$. Thus,

$$\psi(d(H((x_1,1)), H((x_2,1)))) = |x_1 - x_2| \leq |x_1 - x_2| - \frac{|x_1 - x_2|}{2} = \frac{|x_1 - x_2|}{2}.$$

$A = \{(x, 0) : 0 \leq x \leq 1\}$ is closed and bounded in $\mathbb{R}^2$, so it is complete. Also, A_0 is proximally closed in A.
Now,

$$d(N((x,0)), H((y,1))) = d((x,1),(y,0)) = \sqrt{(x-y)^2 + 1}.$$

For $(x, 0) \in A$ and $(y, 1) \in B$, the minimum distance occurs when $x = y$, giving:

$$d(N((x,0)), H((y,1))) = 1 = d(A, B).$$

For any $(x_0, 0) \in A_0$ and $(y_0, 1) \in B_0$, we construct sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$x_{n+1} = N(x_n), \quad \text{starting with } x_0 \in A_0,$$
$$y_{n+1} = H(y_n), \quad \text{starting with } y_0 \in B_0.$$

Since $N((x,0)) = (x,1)$ and $H((x,1)) = (x,0)$, the sequences are:

$$x_n = (u_n, 1) \quad \text{for } n \text{ even and } x_n = (u_n, 0) \text{ for } n \text{ odd,}$$
$$y_n = (u_n, 0) \quad \text{for } n \text{ even and } y_n = (u_n, 1) \text{ for } n \text{ odd.}$$

Thus, the best proximity points are $x = (u,1) \in B$ and $y = (u,0) \in A$. The distance between x and y is:

$$d(x,y) = d((u,1),(u,0)) = 1 = d(A,B).$$

Conclusions and Future Works

In this research work, we established best proximity point results for generalized $\psi - \phi$ contraction mappings and proximal cyclic contraction mappings. We utilize the newly introduced proximal metrical notions such as proximal sequence, proximal completeness and proximal continuity by A. Alam [1]. Moreover, we provide non-trivial examples to substantiate the validity of the main results. These findings unifies and broadens the scope of existing best proximity point theory for non-self mappings on the settings of metric spaces.

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