J. Korean Soc. Math. Edu. Ser. B: Theoretical Math. Pedagogical Math.
https://doi.org/10.7468/jksmeb.2025.32.3.205 ISSN(Print) 3059-0604
Volume 32, Number 3 (August 2025), Pages 205–215 ISSN(Online) 3059-1309

POSITIVE ALMOST PERIODIC SOLUTIONS OF THE MACKEY-GLASS MODEL WITH ITERATIVE TERMS

ShuoQi Wu[a,*], BingWen Liu[b] and JiaNi Wu [c]

Abstract. This paper is devoted to investigate a non-autonomous Mackey-Glass model with multiple iterative terms. We develop novel differential inequality techniques to construct appropriate positive invariant sets, and acquire the existence and uniqueness of positive almost periodic solution for the addressed model, which supplement and extend the corresponding results in existing literature.

1. Introduction

Delay differential equations, as an important class of mathematical models, are prevalently applied in diverse areas including biological sciences, economics, physics, and engineering to describe dynamic systems with time-lag effects. The behavior of these systems often exhibits complex temporal structures, including chaos, periodicity, and almost periodicity. The Mackey-Glass model, a classic nonlinear delay differential equation model, was initially put forward by Mackey and Glass [1] in 1977 with the intention of depicting the hematopoietic process under a single-peak production rate. The specific model can be expressed as follows:

$$u'(t) = -\gamma u(t) + \frac{\beta\theta^n u(t-\tau)}{\theta^n + u^n(t-\tau)}, \tag{1.1}$$

where $n > 0$, $u(t)$ describes the density of mature circulating cells, τ represents the time required for cells produced in the bone marrow to develop from birth to maturity, and γ, β, θ are all positive constants, representing the destruction rate, maximum production rate, and shape parameter, respectively. Subsequently, extensive studies have been carried out on the various dynamics of the model (1.1). In addition, numerous generalizations of the model (1.1) have also been put forward

Received by the editors January 22, 2025. Revised May 22, 2025. Accepted June 24, 2025.
2020 *Mathematics Subject Classification.* 34C27, 34K25.
Key words and phrases. Mackey-Glass model, almost periodic solution, Banach fixed point theorem, iterative term.
*Corresponding author.

[2–7], including models with multiple time-varying delays [4, 5], models with two different distributed delays [6], and the reaction-diffusion models [3, 6, 7], etc. At the same time, various qualitative properties of these models have been researched in depth.

It should be noted that, based on the characteristic that the time delay depends on the state variable, many delayed dynamical models in bioscience are described by iterative differential equations [8-12]. In particular, the model (1.1) can be naturally generalized to the following non-autonomous differential equation with multiple iterative terms involving state-dependent delays:

$$u'(t) = -a(t)u(t) + \sum_{i=1}^{m} b_i(t)\frac{u^{[i]}(t)}{1+(u^{[i]}(t))^q}, \tag{1.2}$$

where $q > 1$ is a positive constant, $a,\ b_i \in C(\mathbb{R}, (0, +\infty))$,

$$u^{[i]}(t) = u(u^{[i-1]}(t)) = u(t - (t - u^{[i-1]}(t)))$$

is an iterative term, $t - u^{[i-1]}(t)$ is the state-dependent delay, $i \in Q := \{1, 2, \cdots, m\}$, and $u^{[0]}(t) = t$. It is worth mentioning that since the iterative differential equation has deviation parameters and the state-dependent delay corresponding to the iteration term depends not only on the time variable but also on the state variable, the dynamical analysis and research of this type of delay differential equation model present many difficulties in mathematical analysis, especially for non-autonomous iterative differential equations with non-constant coefficient parameters, the research results are relatively scarce (see [13]).

In the study of the dynamical behavior of biological mathematical models, it is often necessary to fully consider the changes in the external environment in which the model is located, such as seasonal fluctuations. These changes are generally periodic or almost periodic, and almost periodicity is more common, more accurate, and more practical than periodicity. In particular, recently, Bouakkaz and Khemis [8]employed the Schauder fixed point theorem in order to investigate the existence of positive periodic solutions regarding the generalized Nicholson's blowflies equation featuring iterative terms and state-dependent delay; the existence and uniqueness of positive periodic solutions for the Nicholson's blowflies equation with time-varying delays and quadratic iterative terms were established by Khemis [9] through the application of the Banach fixed point theorem; Liu and Tunç [12] set up the existence of pseudo almost periodic solutions for a class of scalar delay differential equations with quadratic iterative terms, but their conclusions did not involve the positivity of pseudo almost periodic solutions. To our knowledge, no literature has studied the existence of positive almost periodic solutions for the model (1.2) with multiple iterative terms.

According to the above analysis and discussion, the objective of this paper is to build the existence of positive almost periodic solutions for the Mackey-Glass model (1.2) with multiple iterative terms, thereby supplementing and improving the corresponding results in existing literature.

The paper is organized as follows: Section 2 introduces some preliminary notations and lemmas that will be used in subsequent sections. Our main result is presented and validated in Section 3. Finally, Sections 4 and 5 provide some numerical examples and conclusions.

2. Preliminary results

Throughout this paper, let $BC(\mathbb{R},\mathbb{R})$ be the set of bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For any $u \in BC(\mathbb{R},\mathbb{R})$, we define $u^+ := \sup_{t\in\mathbb{R}} |u(t)|$, $u^- := \inf_{t\in\mathbb{R}} |u(t)|$.

Let $f(x) := \frac{x}{1+x^q}$. Then from basic knowledge of mathematical analysis, we know that when $q > 1$, the function f increases on $(0, (\frac{1}{q-1})^{\frac{1}{q}})$ and decreases on $((\frac{1}{q-1})^{\frac{1}{q}}, +\infty)$. It is obvious that the derivative $f'(x) = \frac{1-x^q(q-1)}{(1+x^q)^2}$ is decreasing on the interval $(0, (\frac{q+1}{q-1})^{\frac{1}{q}})$ and increasing on the interval $((\frac{q+1}{q-1})^{\frac{1}{q}}, +\infty)$, and $f'((\frac{q+1}{q-1})^{\frac{1}{q}}) = -\frac{(1-q)^2}{4q}$. In order to construct an appropriate set of positive invariants, we choose two suitable normal numbers K_1 and $\tilde{K}_1$ based on the relationship between $\frac{(1-q)^2}{4q}$ and 1.

Case 1: When $\frac{(q-1)^2}{4q} < 1$, one can select a unique pair of positive numbers $K_1 \in (0, (\frac{1}{q-1})^{\frac{1}{q}})$ and $\tilde{K}_1 \in ((\frac{1}{q-1})^{\frac{1}{q}}, +\infty)$, such that $\sup_{x\geq K_1} \left|\frac{1-(q-1)x^q}{(1+x^q)^2}\right| = \frac{1-(q-1)K_1^q}{(1+K_1^q)^2}$ and $\frac{K_1}{1+K_1^q} = \frac{\tilde{K}_1}{1+\tilde{K}_1^q}$.

Case 2: When $\frac{(q-1)^2}{4q} \geq 1$, one can find $\tilde{K}_1 \in ((\frac{1}{q-1})^{\frac{1}{q}}, +\infty)$ such that

$$\sup_{x\geq \tilde{K}_1} \left|\frac{1-(q-1)x^q}{(1+x^q)^2}\right| = \left|\frac{1-(q-1)\tilde{K}_1^q}{(1+\tilde{K}_1^q)^2}\right|.$$

In addition, there exists $K_1 \in (0, (\frac{1}{q-1})^{\frac{1}{q}})$ satisfying $\frac{\tilde{K}_1}{1+\tilde{K}_1^q} = \frac{K_1}{1+K_1^q}$.

Next, we adopt the following key assumption:

(A_1) There are positive numbers ζ^*, ζ^{**} and K_2, such that $K_2 \in (K_1, \tilde{K}_1)$, $-\zeta^* = \sup\limits_{t\in\mathbb{R}}(-a(t)K_2 + \sum\limits_{i=1}^{m} b_i(t)\frac{1}{q}(q-1)^{1-\frac{1}{q}})$, and $\zeta^{**} = \inf\limits_{t\in\mathbb{R}}(-a(t) + \sum\limits_{i=1}^{m} b_i(t)\frac{1}{1+K_1^q})$.

According to the literature [14], we can obtain the following definition and lemmas.

Definition 2.1 [14] A function $h \in C(\mathbb{R}, \mathbb{R})$ is said to be almost periodic if for any $\varepsilon > 0$, it is possible to find a positive number with the property that in every interval of length $l(\varepsilon)$, there exists a point $\sigma(\varepsilon)$ fulfilling the condition,

$$|h(t+\sigma) - h(t)| < \varepsilon \text{ for all } t \in \mathbb{R}.$$

We denote by $AP(\mathbb{R}, \mathbb{R})$ (resp. $AP(\mathbb{R}, \mathbb{R}+)$) the set of almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$ (resp. $\mathbb{R}+$). In particular, $(AP(\mathbb{R}, \mathbb{R}), \|\cdot\|_\infty)$ is a Banach space, where $\|f\|_\infty := \sup\limits_{t\in\mathbb{R}} |f(t)|$. Besides, assume that a, $b_i \in AP(\mathbb{R}, \mathbb{R}_+)$, $a^- > 0$, where $i \in Q$, and $\mathbb{R}_+ = [0, +\infty)$.

Lemma 2.1 [14] If $A \in AP(\mathbb{R}, \mathbb{R}_+)$ with $\inf\limits_{t\in\mathbb{R}} A(t) > 0$, and $B \in AP(\mathbb{R}, \mathbb{R})$, then there exists a unique almost periodic solution for the non-homogeneous linear differential equation $u'(t) = -A(t)u(t) + B(t)$, which can be expressed as $u(t) = \int_{-\infty}^{t} e^{-\int_s^t A(\tau)d\tau} B(s)ds$.

Lemma 2.2 [14] Assume that $f(t) \in C(\mathbb{R}, \mathbb{R})$ is an almost periodic function. Then $f(t)$ is bounded and uniformly continuous on $\mathbb{R}$.

Lemma 2.3 [14] Let u, $v : \mathbb{R} \to \mathbb{R}$ be almost periodic functions. Then $u \pm v$ and uv are also almost periodic functions, and $\frac{u}{v}$ is also an almost periodic function when $\inf\limits_{t\in\mathbb{R}} |v(t)| > 0$ is satsified. Further, if F is uniformly continuous in the value domain of v, then $F \circ v$ is also an almost periodic function.

3. The existence and uniqueness of positive almost periodic solutions

Theorem 3.1 Assume that (A_1) holds, and

$$\frac{(q-1)^2}{4q} < 1, \ \sup_{t\in\mathbb{R}} \left\{ -a(t) + \sum_{i=1}^{m} \frac{1-M^i}{1-M} b_i(t) \frac{1-K_1^q(q-1)}{(1+K_1^q)^2} \right\} < 0, \tag{3.1}$$

where $M := \sum\limits_{i=1}^{m} b_i^+ K_2 (1 + \frac{a^+}{a^-})$. Then, there exists a unique positive almost periodic solution of (1.2) in the region

$$\Omega := \{\varphi \in AP(\mathbb{R}, \mathbb{R}_+) : K_1 \le \varphi(t) \le K_2, |\varphi(t_2) - \varphi(t_1)| \le M|t_2 - t_1|, t_1, t_2 \in \mathbb{R}\}.$$

Proof. By (3.1), one can choose a constant $\varsigma \in (0,1]$, such that

$$\sup_{t\in\mathbb{R}}\left\{-a(t)+\sum_{i=1}^{m} b_i(t)\ \frac{1-M^i}{1-M}\ \frac{1-K_1^q(q-1)}{(1+K_1^q)^2}e^{\varsigma}\right\}<0.$$

We first show that Ω is a closed set. Indeed, let $\{\varphi_n\} \subseteq \Omega$ and $\lim_{n\to\infty} \|\varphi_n - \varphi\|_\infty = 0$. Then, according to the completeness of $AP(\mathbb{R}, \mathbb{R}_+)$, one can obtain $\varphi \in AP(\mathbb{R}, \mathbb{R}_+)$. Since for any $t \in \mathbb{R}$, we have $\varphi_n(t) \in [K_1, K_2]$, and $\{\varphi_n\}$ converges pointwise on $\mathbb{R}$, $\varphi(t) \in [K_1, K_2]$. Moreover, for arbitrary $t_1, t_2 \in \mathbb{R}$, $|\varphi_n(t_2) - \varphi_n(t_1)| \le M|t_2 - t_1|$, thereupon

$$|\varphi(t_2)-\varphi(t_1)| \le M|t_2-t_1|.$$

Thus, Ω is a closed set.

Secondly, we shall define an abstract operator T. Indeed, according to Lemmas 2.2 and 2.3, we acquire

$$\sum_{i=1}^{m} b_i(t)\frac{\varphi^{[i]}(t)}{1+(\varphi^{[i]}(t))^q} \in AP(\mathbb{R},\mathbb{R}_+) \text{ for any } \ \varphi\in\Omega.$$

In view of the fact that $a^- > 0$, it follows from Lemma 2.1 that the auxiliary equation

$$u'(t) = -a(t)u(t) + \sum_{i=1}^{m} b_i(t)\frac{\varphi^{[i]}(t)}{1+(\varphi^{[i]}(t))^q},\ \ t\in\mathbb{R},$$

has exactly one almost periodic solution:

$$u^{\varphi}(t) = \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}\left[\sum_{i=1}^{m} b_i(s)\frac{\varphi^{[i]}(s)}{1+(\varphi^{[i]}(s))^q}\right]ds. \tag{3.2}$$

Then, we define a mapping $T : \Omega \to AP(\mathbb{R}, \mathbb{R}_+)$ by setting $(T\varphi)(t) = u^\varphi(t)$ for any $\varphi \in \Omega$ and $t \in \mathbb{R}$.

Furthermore, we show that Ω has a positive invariance under the abstract operator T. In truth, for any $\varphi \in \Omega$, in view of (3.2), (A_1) and the fact that $\sup_{u\ge 0} \frac{u}{1+u^q} = \frac{1}{q}(q-1)^{1-\frac{1}{q}}$, we obtain

$$\begin{aligned} u^{\varphi}(t) &\le \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}\left[\sum_{i=1}^{m} b_i(s)\frac{1}{q}(q-1)^{1-\frac{1}{q}}\right]ds \\ &\le \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}(-\zeta^* + a(s)K_2)ds \\ &\le \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}a(s)K_2 ds = K_2,\ \ t\in\mathbb{R}. \end{aligned}$$

According to (A_1) and $\min\limits_{u\in[K_1,\tilde{K}_1]} \frac{u}{1+u^q} = \frac{K_1}{1+K_1^q}$, (3.2) yields

$$\begin{aligned} u^{\varphi}(t) &\geq \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau} \left[\sum_{i=1}^{m} b_i(s)\frac{K_1}{1+K_1^q}\right] ds \\ &\geq \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}(\zeta^{**}+a(s))K_1 ds \\ &\geq \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau} a(s)K_1 ds = K_1,\ t\in\mathbb{R}, \end{aligned}$$

which leads to $K_1 \leq u^{\varphi}(t) \leq K_2$.

To continue our argument, we state that $|u^{\varphi}(t_2)-u^{\varphi}(t_1)| \leq M\,|t_2-t_1|$ for any $t_1,\ t_2\in\mathbb{R}$, and $\varphi\in\Omega$. Assume $t_2>t_1$ (The situation is similar to $t_2<t_1$.). Then, from

$$\begin{aligned} &|u^{\varphi}(t_2)-u^{\varphi}(t_1)| \\ &\leq \int_{t_1}^{t_2} \left| e^{-\int_s^{t_2} a(\tau)d\tau} \left[\sum_{i=1}^{m} b_i(s)\frac{\varphi^{[i]}(s)}{1+(\varphi^{[i]}(s))^q}\right]\right| ds \\ &\quad + \int_{-\infty}^{t_1} \left| e^{-\int_s^{t_2} a(\tau)d\tau} - e^{-\int_s^{t_1} a(\tau)d\tau}\right| \times \left|\sum_{i=1}^{m} b_i(s)\frac{\varphi^{[i]}(s)}{1+(\varphi^{[i]}(s))^q}\right| ds \\ &\leq \sum_{i=1}^{m} b_i^+ K_2\,|t_2-t_1| + \int_{-\infty}^{t_1} \sum_{i=1}^{m} b_i^+ K_2 \left| e^{-\int_s^{t_2} a(\tau)d\tau} - e^{-\int_s^{t_1} a(\tau)d\tau}\right| ds, \end{aligned}$$

and

$$|e^{-\int_s^{t_2} a(\tau)d\tau} - e^{-\int_s^{t_1} a(\tau)d\tau}| \leq e^{-\int_s^{t_1} a(\tau)d\tau} a^+ |t_2-t_1| \leq a^+ e^{-a^-(t_1-s)}|t_2-t_1|,$$

we get

$$|u^{\varphi}(t_2)-u^{\varphi}(t_1)| \leq \sum_{i=1}^{m} (1+\frac{a^+}{a^-}) b_i^+ K_2\,|t_2-t_1| = M\,|t_2-t_1|\,.$$

This reveals the above statement, and Ω possesses positive invariance under the abstract operator T with $T\Omega\subseteq\Omega$.

In the following, our goal is to verify that $T:\Omega\to\Omega$ is a contraction operator. To do this, one can easily reveal that, for any $\varphi,\psi\in\Omega$,

$$\begin{aligned} &\left\| u^{\varphi} - u^{\psi} \right\|_{\infty} \\ &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau} \left| \sum_{i=1}^{m} b_i(s) \left(\frac{\varphi^{[i]}(s)}{1+(\varphi^{[i]}(s))^q} - \frac{\psi^{[i]}(s)}{1+(\psi^{[i]}(s))^q} \right) \right| ds, \end{aligned} \tag{3.3}$$

and for any $i \in Q$,

$$\begin{aligned}
&\left|\varphi^{[i]}(t) - \psi^{[i]}(t)\right| \\
\leq &\Big[\left|\varphi^{[i-1]}(\varphi(t)) - \varphi^{[i-1]}(\psi(t))\right| + \left|\varphi^{[i-1]}(\psi(t)) - \varphi^{[i-2]}(\psi^2(t))\right| \\
&+ \left|\varphi^{[i-2]}(\psi^2(t)) - \varphi^{[i-3]}(\psi^3(t))\right| + \cdots + \left|\varphi(\psi^{[i-1]}(t)) - \psi(\psi^{[i-1]}(t))\right|\Big] \\
\leq &\frac{1-M^i}{1-M}\|\varphi - \psi\|_\infty.
\end{aligned} \tag{3.4}$$

Note that $\frac{1-K_1^q(q-1)}{(1+K_1^q)^2} = \sup\limits_{x\geq K_1}\left|\frac{1-x^q(q-1)}{(1+x^q)^2}\right|$, and

$$\left|\frac{x}{1+x^q} - \frac{y}{1+y^q}\right| = \left|\frac{1-[x+\theta(y-x)]^q(q-1)}{\{1+[x+\theta(y-x)]^q\}^2}\right| |x-y| \leq \frac{1-K_1^q(q-1)}{(1+K_1^q)^2}|x-y|,$$

where $x,\, y \in [K_1, +\infty)$, $0 < \theta < 1$. (3.3) and (3.4) suggest that

$$\begin{aligned}
&\left\|u^\varphi - u^\psi\right\|_\infty \\
\leq &\sup_{t\in\mathbb{R}}\int_{-\infty}^t e^{-\int_s^t a(\tau)d\tau}\sum_{i=1}^m b_i(s)\frac{1-K_1^q(q-1)}{(1+K_1^q)^2}\left|\varphi^{[i]}(s) - \psi^{[i]}(s)\right|ds \\
\leq &\sup_{t\in\mathbb{R}}\int_{-\infty}^t e^{-\int_s^t a(\tau)d\tau}\sum_{i=1}^m \frac{1-M^i}{1-M}\|\varphi-\psi\|_\infty \times b_i(s)\frac{1-K_1^q(q-1)}{(1+K_1^q)^2}ds \\
\leq &\|\varphi-\psi\|_\infty \sup_{t\in\mathbb{R}}\int_{-\infty}^t e^{-\int_s^t a(\tau)d\tau}a(s)\frac{1}{e^\varsigma}ds \\
\leq &\frac{1}{e^\varsigma}\|\varphi-\psi\|_\infty.
\end{aligned}$$

Therefore, $\frac{1}{e^\varsigma} < 1$ implies that $T : \Omega \to \Omega$ is a contraction mapping.

In summary, combining the Banach fixed point theorem, we can get that T possesses a unique fixed point $\varphi(t)$ in Ω. Namely, $\varphi(t)$ is a positive almost periodic solution of (1.2) in Ω. The proof of Theorem 3.1 is now completed.

Theorem 3.2 Assume that (A_1) holds, and

$$\frac{(q-1)^2}{4q} \geq 1, \quad \sup_{t\in\mathbb{R}}\left\{-a(t) + \sum_{i=1}^m \frac{1-M^i}{1-M}b_i(t)\left|\frac{1-\tilde{K}_1^q(q-1)}{(1+\tilde{K}_1^q)^2}\right|\right\} < 0. \tag{3.5}$$

Then (1.2) has a unique positive almost periodic solution in Ω.

Proof. Based on (3.5), it is evident that there exists a constant $\tilde{\varsigma} \in (0,1]$, such that

$$\sup_{t\in\mathbb{R}}\left\{-a(t) + \sum_{i=1}^m b_i(t)\,\frac{1-M^i}{1-M}\left|\frac{1-\tilde{K}_1^q(q-1)}{(1+\tilde{K}_1^q)^2}\right| e^{\tilde{\varsigma}}\right\} < 0.$$

Similar to Theorem 3.1, one can easily verify that Ω is a closed set, and Ω has positive invariance under the abstract operator T. Next, we demonstrate that T is a contraction mapping on Ω. Note that, when $\frac{(q-1)^2}{4q} \geq 1$, $\sup\limits_{x\geq \tilde{K}_1}\left|\frac{1-(q-1)x^q}{(1+x^q)^2}\right| = \left|\frac{1-(q-1)\tilde{K}_1^q}{(1+\tilde{K}_1^q)^2}\right|$, and

$$\left|\frac{x}{1+x^q}-\frac{y}{1+y^q}\right| = \left|\frac{1-[x+\theta(y-x)]^q(q-1)}{\{1+[x+\theta(y-x)]^q\}^2}\right||x-y| \leq \left|\frac{1-\tilde{K}_1^q(q-1)}{(1+\tilde{K}_1^q)^2}\right||x-y|,$$

where $x,\, y \in [\tilde{K_1}, +\infty)$, $0<\theta<1$. Then, by (3.3) and (3.4), we get

$$\begin{aligned}
&\left\|u^{\varphi}-u^{\psi}\right\|_\infty \\
\leq& \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}\sum_{i=1}^{m} b_i(s)\left|\frac{1-\tilde{K}_1^q(q-1)}{(1+\tilde{K}_1^q)^2}\right|\left|\varphi^{[i]}(s)-\psi^{[i]}(s)\right|ds \\
\leq& \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}\sum_{i=1}^{m}\frac{1-M^i}{1-M}\|\varphi-\psi\|_\infty \times b_i(s)\left|\frac{1-\tilde{K}_1^q(q-1)}{(1+\tilde{K}_1^q)^2}\right|ds \\
\leq& \|\varphi-\psi\|_\infty \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\int_s^t a(\tau)d\tau}a(s)\frac{1}{e^{\tilde{\varsigma}}}ds \\
\leq& \frac{1}{e^{\tilde{\varsigma}}}\|\varphi-\psi\|_\infty.
\end{aligned}$$

Therefore, $T:\Omega\to\Omega$ is a contraction operator, and we know that T possesses a unique fixed point $\varphi(t)$ in Ω. Namely, $\varphi(t)$ is a positive almost periodic solution of (1.2) in Ω. The proof of Theorem 3.2 is now completed.

Remark 3.1 Since the existing literature does not address the research on the existence and uniqueness of positive almost periodic solutions of (1.2) with iterative terms, the results in Theorem 3.1 and Theorem 3.2 of this paper are completely new. To a certain extent, they improve the relevant conclusions in [8-12].

4. Simulation examples

Example 4.1 Let $q=2$, $m=2$, $a\equiv 0.058$, and $b_1(t)=b_2(t)=\frac{1}{20}\cdot\frac{200+\cos\sqrt{2}t}{200+\sin t}$. In this case, $\frac{(q-1)^2}{4q}=\frac{1}{8}<1$. Consider the following Mackey-Glass model with iterative terms:

$$\begin{aligned}
x'(t) = -0.058x(t) &+ \frac{1}{20}\cdot\frac{200+\cos\sqrt{2}t}{200+\sin t}\frac{x(t)}{1+(x(t))^2} \\
&+ \frac{1}{20}\cdot\frac{200+\cos\sqrt{2}t}{200+\sin t}\frac{x^{[2]}(t)}{1+(x^{[2]}(t))^2}.
\end{aligned} \tag{4.1}$$

Through simple numerical calculations, we can obtain that $b_i^+ = \frac{1}{20}\cdot\frac{201}{199}$, $b_i^- = \frac{1}{20}\cdot\frac{199}{201}$, $K_1 \approx 0.8105, \tilde{K}_1 \approx 1.2309$. Take $K_2 = 0.9$. Then $K_2 > K_1 > 0$, $M \approx 0.1818$,$-\zeta^* \approx -0.0017$, and $\zeta^{**} \approx 0.0018$. Thus,

$$\sup_{t\in\mathbb{R}}\left\{-a(t) + \sum_{i=1}^{m}\frac{1-M^i}{1-M}b_i(t)\frac{1-K_1^q(q-1)}{(1+K_1^q)^2}\right\} \approx -0.04423 < 0.$$

In summary, the model (4.1) satisfies all the conditions of Theorem 3.1. Consequently, for the model (4.1), there exists a unique positive almost periodic solution in the set

$$\Omega = \{\psi \in AP(\mathbb{R},\mathbb{R}_+) : 0.8105 \le \psi(t) \le 0.9, |\psi(t_2)-\psi(t_1)| \le 0.1818|t_2-t_1|, t_1, t_2 \in \mathbb{R}\}.$$

Example 4.2 Let $q = 6$, $m = 2$, $a \equiv 0.25$, and $b_1(t) = b_2(t) = \frac{1}{20}\cdot\frac{200+\cos\sqrt{2}t}{200+\sin t}$. At this time, $\frac{(q-1)^2}{4q} = \frac{25}{24} > 1$. Consider the following Mackey-Glass model with iterative terms:

$$\begin{aligned} x'(t) = &- 0.25x(t) + \frac{1}{20}\cdot\frac{200+\cos\sqrt{2}t}{200+\sin t}\frac{x(t)}{1+(x(t))^6} \\ &+ \frac{1}{20}\cdot\frac{200+\cos\sqrt{2}t}{200+\sin t}\frac{x^{[2]}(t)}{1+\left(x^{[2]}(t)\right)^6}. \end{aligned} \tag{4.2}$$

Through simple numerical calculations, we can obtain that $b_i^+ = \frac{1}{20}\cdot\frac{201}{199}, b_i^- = \frac{1}{20}\cdot\frac{199}{201}, K_1 \approx 0.4441, \tilde{K}_1 \approx 1.0577$. Take $K_2 = 0.8$. Then $K_2 > K_1 > 0$, $M \approx 0.1616$, $-\zeta^* \approx -0.0069$, and $\zeta^{**} \approx 0.0448$. Consequently,

$$\sup_{t\in\mathbb{R}}\left\{-a(t) + \sum_{i=1}^{m}\frac{1-M^i}{1-M}b_i(t)\left|\frac{1-\tilde{K}_1^q(q-1)}{\left(1+\tilde{K}_1^q\right)^2}\right|\right\} \approx -0.1363 < 0.$$

In conclusion, (4.2) satisfies all the assumptions of Theorem 3.2. Therefore, (4.2) has a unique positive almost periodic solution in the set

$$\Omega = \{\psi \in AP(\mathbb{R},\mathbb{R}_+) : 0.4441 \le \psi(t) \le 0.8, |\psi(t_2)-\psi(t_1)| \le 0.1616|t_2-t_1|, t_1, t_2 \in \mathbb{R}\}.$$

5. Conclusions

In this paper, by considering the influence of state-dependent delays of iterative terms, a non-autonomous Mackey-Glass model with multiple iterative terms is proposed. By applying the differential inequality technique and the Banach fixed point theorem, we prove the existence and uniqueness of positive almost periodic solutions of the considered model on an appropriate positive invariant set. Meanwhile, we provide numerical simulation data, which strongly verify the correctness and effectiveness of the obtained theoretical results. It is worth mentioning that the research method of this paper can be used to explore the problem of positive almost-periodic solutions of other nonlinear biological models with state-dependent delays and multiple iterative terms, which will be the main content of our future research.

References

1. M. Mackey, L. Glass: Oscillation and chaos in physiological control systems. *Science* 197(1977), 287–289. `DOI: 10.1126/science.26732`
2. D. Jiang, J. Wei & B. Zhang: Positive periodic solutions of functional differential equations and population model. *Electronic Journal of Differential Equations* 17 (2002), Paper No. 71.
3. C. Ramirez-Carrasco, J. Molina-Garay : Existence and approximation of traveling wavefronts for the diffusive Mackey-Glass equation. *Australian Journal of Mathematical Analysis and Applications* 18 (2021), no. 1, Art. No. 2.
4. Y. Tan: Dynamics analysis of Mackey-Glass model with two variable delays. *Mathematical Biosciences and Engineering* 17 (2020): 4513-4526. `Doi: 10.3934/mbe.2020249`
5. L. Berezansky, E. Braverman: A note on stability of Mackey-Glass equations with two delays. *Journal of Mathematical Analysis and Applications* 450 (2017): 1208-1228. `https://doi.org/10.1016/j.jmaa.2017.01.050`
6. C. Huang, X. Ding: Dynamics of the diffusive Nicholson's blowflies equation with two distinct distributed delays. *Applied Mathematics Letters* 145 (2023): 108741. `https://doi.org/10.1016/j.aml.2023.108741`
7. C. Huang, X. Ding: Existence of traveling wave fronts for a diffusive Mackey-Glass model with two delays. *Nonlinear Analysis: Real World Applications* 76 (2024): 104024. `https://doi.org/10.1016/j.nonrwa.2023.104024`
8. A. Bouakkaz, R. Khemis: Positive periodic solutions for revisited Nicholson's blowflies equation with iterative harvesting term. *Journal of Mathematical Analysis and Applications* 449 (2021): 124663. `https://doi.org/10.1016/j.jmaa.2020.124663`
9. R. Khemis: Existence, uniqueness and stability of positive periodic solutions for an iterative Nicholson's blowflies equation. *Journal of Applied Mathematics and Computing* 69 (2023): 1903-1916. `https://doi.org/10.1007/s12190-022-01820-0`
10. A. Bouakkaz: Positive periodic solutions for a class of first-order iterative differential equations with an application to a hematopoiesis model. *Carpathian Journal of Mathematics* 38 (2022): 347-355. `DOI: https://doi.org/10.37193/CJM.2022.02.07`

11. M. Khemis, A. Bouakkaz & R. Khemis: Existence, uniqueness and stability results of an iterative survival model of red blood cells with a delayed nonlinear harvesting term. *Journal of Mathematical Modeling* 10 (2022): 515-528. DOI: 10.22124/JMM.2022.21577.1892
12. B. Liu, C. Tunç: Pseudo almost periodic solutions for a class of first order differential iterative equations. *Applied Mathematics Letters* 40 (2015): 29-34. https://doi.org/10.1016/j.aml.2014.08.019
13. F. Hartung, T. Krisztin, H.O. Walther & J. Wu: Functional differential equations with state-dependent delays: theory and applications. *Handbook of Differential Equations: Ordinary Differential Equations.* 2006. https://doi.org/10.1016/S1874-5725(06)80009-X
14. A.M. Fink: *Almost periodic differential equations.* Springer, Berlin, 1974.

[a]Research Scholar: College of Data Science, Jiaxing University, Jiaxing, 314001, P. R. China
Email address: wushuoqi2025@126.com

[b]Research Scholar: College of Data Science, Jiaxing University, Jiaxing, 314001, P. R. China
Email address: liubw007@aliyun.com

[c]Research Scholar: School of mathematics and statistics, Changsha University of Science And Technology, Changsha, 410114, P. R. China
Email address: jianiw22@163.com