J. Korean Soc. Math. Edu. Ser. B: Theoretical Math. Pedagogical Math.
https://doi.org/10.7468/jksmeb.2025.32.3.191 ISSN(Print) 3059-0604
Volume 32, Number 3 (August 2025), Pages 191–203 ISSN(Online) 3059-1309

AN INVESTIGATION OF ELEGANT FUZZY LABELING ON STAR GRAPHS AND THEIR VARIANTS

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Abstract. Graph labeling under various constraints is crucial for modeling complex systems, allowing different levels of information precision. Elegant Fuzzy Labeling, a flexible fuzzy labeling model, enhances adaptability over traditional methods. It accommodates diverse labeling approaches for different translation constants, broadening its applicability. In this paper, we extend Elegant Fuzzy Labeling to star graphs and their variants. We prove that all star graphs and the join of two star graphs admit this labeling. Additionally, we establish conditions for bistar graphs to admit this labeling, paving the way for further research in this domain.

1. Introduction

Fuzzy graph theory is a powerful extension of classical graph theory designed to handle uncertainty and imprecision in complex systems. Fuzzy labeling, a key concept of fuzzy graph theory, was introduced by Gani et al. [5, 9]. Fuzzy labeling is an extension of traditional labeling in graph theory that allows for a more flexible and nuanced representation of uncertainties. The exploration of fuzzy labeling in graph theory has become a focal point for many researchers. Notably, [1, 3, 4, 5, 7, 8, 11, 13, 14, 15, 16, 17] underscores the significance of this research direction.

In a previous paper [10], we defined a new type of fuzzy labeling called Elegant Fuzzy Labeling (μ, ρ) as follows: for a given graph G with q edges, the vertex labeling function μ maps $V(G)$ to the interval $[0, 1]$, with $\mu(v) = f(v).h$, where f is an injective function from $V(G)$ into the set $\{h', h'+1, \ldots, h'+q-1, h'+q\}$ and h' is a suitable constant. Similarly, the edge labeling function ρ maps $E(G)$ to $[0, 1]$, with $\rho(uv) = g(uv).h$, where g is an injective function from $E(G)$ into the set $\{0, 1, \ldots, q\}$ defined as $g(uv) = (f(u)+f(v)) \bmod (q+1)$ for every uv in $E(G)$, and $0 \le h \le \frac{1}{h'+q}$

Received by the editors December 8, 2024. Revised February 22, 2025. Accepted March 31, 2025.
2020 *Mathematics Subject Classification.* 05C78, 05C72.
Key words and phrases. Elegant labeling, Fuzzy graph, Fuzzy labeling, Elegant Fuzzy Labeling.
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such that the edge labels are distinct and non-zero, and $\rho(uv) < \mu(u) \wedge \mu(v)$ for all $uv \in E(G)$.

In our previous work [10], we used the translation constant $h' = q + 1$ and the contraction scale $h = \frac{1}{2q+1}$ and investigated the path graphs P_n, $n \neq 4$, cycles C_n, $n \equiv 0, 3 \bmod 4$, and their respective line graphs $L(P_n)$, $n \neq 5$ and $L(C_n)$, $n \equiv 0, 3 \bmod 4$, proving that they admit Elegant Fuzzy Labeling.

In this paper, we extend our investigation to the star graphs and some of its variants, using the same translation constant $h' = q + 1$ and the contraction scale $h = \frac{1}{2q+1}$. We prove that the star graphs S_n, which are trees on n vertices with one vertex having vertex degree $n - 1$ and the remaining $n - 1$ vertices having vertex degree 1, admit Elegant Fuzzy Labeling for every n. Additionally, we also prove that the bistar graphs $B_{2n+1,m}$, where $n = 1, 2, \ldots, m = 2, 3, \ldots$, admit Elegant Fuzzy Labeling. Further we explore the join of two disjoint graphs G and H, denoted $G + H$, which is the graph obtained from G and H by joining each vertex of G to every vertex of H [6]. In particular, we prove that the join of two disjoint star graphs $S_m + S_n$, admit Elegant Fuzzy Labeling. By doing so, we provide new insights into its broader applicability and contribute to the development of more nuanced fuzzy labeling techniques. Throughout this paper, the vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$. The overall structure of this study is illustrated in the following flow chart (Fig.1) for clarity.

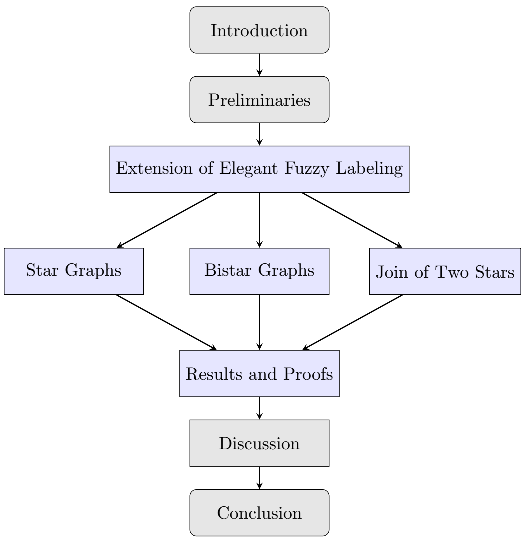

Figure 1

2. Preliminaries

In this section, we review some definitions from sources [2, 5, 10, 12], which are essential for subsequent discussion.

Definition 2.1. Elegant labeling f of a graph G with q edges is an injective function from the $V(G)$ into the set $\{0, 1, \ldots, q\}$ such that the function g from $E(G)$ into the set $\{1, 2, \ldots, q\}$ defined as $g(uv) = (f(u) + f(v)) \bmod (q + 1)$ for every uv in $E(G)$ is injective.

Definition 2.2. A graph $G = (\mu, \rho)$ with vertex set $V(G)$, edge set $E(G)$, and a pair of functions $\mu : V(G) \to [0, 1]$ and $\rho : E(G) \to [0, 1]$ is called a fuzzy graph if for every uv in $E(G)$, $\rho(uv) \leq \mu(u) \wedge \mu(v)$.

Definition 2.3. A fuzzy graph $G = (\mu, \rho)$ is said to be a fuzzy labeling graph, if $\mu : V(G) \to [0, 1]$ and $\rho : E(G) \to [0, 1]$ are injective and for every uv in $E(G)$, $\rho(uv) < \mu(u) \wedge \mu(v)$.

Definition 2.4. Let G be a (p, q) graph. Let f be an injective function from $V(G)$ into the set $\{q + 1, q + 2, \ldots, 2q + 1\}$, and g be an injective function from $E(G)$ into the set $\{0, 1, \ldots, q\}$ defined as, $g(uv) = (f(u) + f(v)) \bmod (q + 1)$, for every $uv \; in \; E(G)$. Define vertex labeling $\mu : V(G) \to [0, 1]$ as $\mu(v) = f(v).h$ and the edge labeling $\rho : E(G) \to [0, 1]$ as $\rho(uv) = g(uv).h$, where $h = \frac{1}{2q+1}$. If the edge labels are distinct and nonzero, and if the condition $\rho(uv) < \mu(u) \wedge \mu(v)$ is satisfied for all $uv \in E(G)$, then G is said to be an Elegant Fuzzy Labeling Graph and (μ, ρ) is an Elegant Fuzzy Labeling of G.

3. Extension of Elegant Fuzzy Labeling to Star Graphs and Their Variants

In this section, we extend Elegant Fuzzy Labeling to star graphs and their variants and provide the corresponding proofs.

Theorem 3.1. *Star graphs S_n admit Elegant Fuzzy Labeling for every n.*

Proof. Let S_n be a star with n vertices, $n-1$ edges, with vertex set $V(S_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E(S_n) = \{v_0v_i, 1 \leq i \leq n-1\}$. Let $f : V(S_n) \to \{n,\ n+1,\ n+2,\ \ldots, 2n-1\}$ be an injective function defined as,

$$f(v_0) = n,$$

$$f(v_i) = n+i,\ 1 \leq i \leq n-1.$$

Let g be a function from $E(S_n)$ into the set $\{0, 1, \ldots, n-1\}$ defined as, $g(v_0v_i) = (f(v_0) + f(v_i)) \bmod n$ for $1 \leq i \leq n-1$. Thus, for $1 \leq i \leq n-1$, we have

$$\begin{aligned} g(v_0v_i) &= (f(v_0) + f(v_i)) \bmod n \\ g(v_0v_i) &= (n+n+i) \bmod n \\ g(v_0v_i) &= i. \end{aligned}$$

Now we have $h = \frac{1}{2n-1}$. The vertex labeling $\mu : V(S_n) \to [0,1]$ is defined as $\mu(v_i) = f(v_i).\frac{1}{2n-1}$ for every $0 \leq i \leq n-1$ and the edge labeling $\rho : E(S_n) \to [0,1]$ is defined as $\rho(v_0v_i) = g(v_0v_i).\frac{1}{2n-1}$ for every $1 \leq i \leq n-1$. It is evident that $g(v_0v_i)$ are distinct and nonzero, therefore $\rho(v_0v_i)$ are distinct and nonzero. Further $\max\{g(v_0v_i)\} = n-1$ and $\min\{f(v_i)\} = n$. Therefore, $g(v_0v_i) < f(v_0) \wedge f(v_i)$ for every $0 \leq i \leq n-1$. Hence for every $1 \leq i \leq n-1$

$$\begin{aligned} \rho(v_0v_i) &= g((v_0v_i).\frac{1}{2n-1} \\ &< (f(v_0) \wedge f(v_i)).\frac{1}{2n-1} \\ &< f(v_0).\frac{1}{2n-1} \wedge f(v_i).\frac{1}{2n-1} \\ &< \mu(v_0) \wedge \mu(v_i). \end{aligned}$$

This implies that, the edge labels are less than the minimum of their respective endpoints' labels. Consequently, S_n admit Elegant Fuzzy Labeling and (μ, ρ) is an Elegant Fuzzy Labeling of S_n. □

Example 3.2. To illustrate Theorem 3.1, we consider a star graph S_6 in Fig.2. By the theorem, an Elegant Fuzzy Labeling (μ, ρ) of S_6 is given as follows: $\mu(v_0) = 0.55$, $\mu(v_1) = 0.64$, $\mu(v_2) = 0.73$, $\mu(v_3) = 0.82$, $\mu(v_4) = 0.91$, $\mu(v_5) = 1$, $\rho(v_0v_1) = 0.09$, $\rho(v_0v_2) = 0.18$, $\rho(v_0v_3) = 0.27$, $\rho(v_0v_4) = 0.36$, $\rho(v_0v_5) = 0.45$. From this, we observe that all edge labels are distinct, nonzero, and strictly less than the labels of their respective endpoints. Hence, S_6 admits Elegant Fuzzy Labeling, as established by Theorem 3.1.

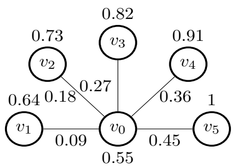

Figure 2. S_6

Theorem 3.3. *Bistar graphs* $B_{2n+1,m}$ *where* $n = 1, 2, 3, \ldots,\ m = 2, 3, \ldots$ *admit Elegant Fuzzy Labeling.*

Proof. Let $B_{2n+1,m}$ where $n = 1, 2, 3, \ldots,\ m = 2, 3, \ldots$ be a bistar with $2n + m + 1$ vertices, $2n+m$ edges, with vertex set $V(B_{2n+1,m}) = \{v_0, v_1, \ldots, v_{2n}, u_0, u_1, \ldots, u_{m-1}\}$ and edge set $E(B_{2n+1,m}) = \{v_0u_0, v_0v_i, u_0u_j; 1 \leq i \leq 2n, 1 \leq j \leq m-1\}$. Let $f : V(B_{2n+1,m}) \to \{2n + m + 1,\ 2n + m + 2, \ldots, 4n + 2m + 1\}$ be an injective function defined as,

$$f(v_i) = \begin{cases} 3n + m + 2 & i = 0 \\ 3n + 2m + 2 & i = 1 \\ f(v_{i-1}) + 1 & 2 \leq i \leq n \\ 2n + m + 1 & i = n + 1 \\ f(v_{i-1}) + 1 & n + 2 \leq i \leq 2n, \end{cases}$$

$$f(u_j) = \begin{cases} 3n + m + 1 & j = 0 \\ 3n + m + 3 & j = 1 \\ f(u_{j-1}) + 1 & 2 \leq j \leq m - 1. \end{cases}$$

Let g be a function from $E(B_{2n+1,m})$ into the set $\{0, 1, \ldots, 2n + m\}$ defined as,

$$g(v_0u_0) = (f(v_0) + f(u_0)) \bmod (2n + m + 1),$$

$$g(v_0v_i) = (f(v_0) + f(v_i)) \bmod (2n + m + 1),\ 1 \leq i \leq 2n,$$

$$g(u_0u_j) = (f(u_0) + f(u_j)) \bmod (2n + m + 1),\ 1 \leq j \leq m - 1.$$

Now, for $i = 0,\ j = 0$,

$$\begin{aligned} g(v_0u_0) &= (f(v_0) + f(u_0)) \bmod (2n + m + 1) \\ &= (3n + m + 2 + 3n + m + 1) \bmod (2n + m + 1) \\ &= (2(2n + m + 1) + 2n + 1) \bmod (2n + m + 1) \\ &= 2n + 1. \end{aligned}$$

For $1 \leq i \leq n$,

$$\begin{aligned} g(v_0v_i) &= (f(v_0) + f(v_i)) \bmod (2n+m+1) \\ &= (3n+m+2+f(v_{i-1})+1) \bmod (2n+m+1) \\ &= (3n+m+2+f(v_{i-2})+2) \bmod (2n+m+1) \\ &\ \vdots \\ &= (3n+m+2+f(v_{i-l})+l) \bmod (2n+m+1), \quad l = 0,1,\ldots,i-1, \end{aligned}$$

and when $l = i-1$,

$$\begin{aligned} g(v_0v_i) &= (3n+m+2+f(v_1)+(i-1) \bmod (2n+m+1) \\ &= (3n+m+2+3n+2m+2+i-1) \bmod (2n+m+1) \\ &= (6n+3m+3+i) \bmod (2n+m+1) \\ &= i. \end{aligned}$$

For $n+1 \leq i \leq 2n$,

$$\begin{aligned} g(v_0v_i) &= (f(v_0) + f(v_i)) \bmod (2n+m+1) \\ &= (3n+m+2+f(v_{i-1})+1) \bmod (2n+m+1) \\ &= (3n+m+2+f(v_{i-2})+2) \bmod (2n+m+1) \\ &\ \vdots \\ &= (3n+m+2+f(v_{i-l})+l) \bmod (2n+m+1),\ l = 0,1,\ldots,i-n-1, \end{aligned}$$

and when $l = i-n-1$,

$$\begin{aligned} g(v_0v_i) &= (3n+m+2+f(v_{n+1})+(i-n-1) \bmod (2n+m+1) \\ &= (3n+m+2+2n+m+1+i-n-1) \bmod (2n+m+1) \\ &= (4n+2m+2+i) \bmod (2n+m+1) \\ &= i. \end{aligned}$$

For $1 \leq j \leq m-1$,

$$\begin{aligned} g(u_0u_j) &= (f(u_0) + f(u_j)) \bmod (2n+m+1) \\ &= (3n+m+1+f(u_{j-1})+1) \bmod (2n+m+1) \\ &= (3n+m+1+f(u_{j-2})+2) \bmod (2n+m+1) \\ &\ \ \vdots \\ &= (3n+m+1+f(u_{j-l})+l) \bmod (2n+m+1),\ l = 0,1,\ldots,j-1, \end{aligned}$$

and when $l = j-1$,

$$\begin{aligned} g(u_0u_j) &= (3n+m+1+f(u_1)+j-1) \bmod (2n+m+1) \\ &= (3n+m+1+3n+m+3+j-1) \bmod (2n+m+1) \\ &= (2(2n+m+1)+2n+1+j) \bmod (2n+m+1) \\ &= 2n+1+j. \end{aligned}$$

Hence,

$$g(v_0u_0) = 2n+1,$$

$$g(v_0v_i) = i,\ 1 \leq i \leq 2n,$$

$$g(u_0u_j) = 2n+1+j,\ 1 \leq j \leq m-1.$$

Now we have $h = \frac{1}{4n+2m+1}$. The vertex labeling $\mu : V(B_{2n+1,m}) \to [0,1]$ is defined as, $\mu(v_i) = f(v_i).\frac{1}{4n+2m+1}$ for every $0 \leq i \leq 2n$ and $\mu(u_j) = f(u_j).\frac{1}{4n+2m+1}$ for every $0 \leq j \leq m-1$, and edge labeling $\rho : E(B_{2n+1,m}) \to [0,1]$ is defined as, $\rho(v_0u_0) = g(v_0u_0).\frac{1}{4n+2m+1}$, $\rho(v_0v_i) = g(v_0v_i).\frac{1}{4n+2m+1}$ and $\rho(u_0u_j) = g(u_0u_j).\frac{1}{4n+2m+1}$ for every $1 \leq i \leq 2n,\ 1 \leq j \leq m-1$.

It is evident that the sets $\{g(v_0u_0)\}$, $\{g(v_0v_i),\ 1 \leq i \leq 2n\}$ and $\{g(u_0u_j),\ 1 \leq j \leq m-1\}$ are mutually disjoint and the elements are distinct and nonzero, therefore edge labels are distinct and nonzero. Further $\max\{g(v_0u_0), g(v_0v_i), g(u_0u_j)\} = 2n+m$ and $\min\{f(v_0), f(u_0), f(v_i), f(u_j)\} = 2n+m+1$. Therefore, $g(v_0u_0) < f(v_0) \wedge f(u_0)$, $g(v_0v_i) < f(v_0) \wedge f(v_i)$ for every $1 \leq i \leq 2n$ and $g(u_0u_j) < f(u_0) \wedge f(u_j)$ for every $1 \leq j \leq m-1$.

Hence, the edge labels are less than the minimum of their respective endpoints' labels. Consequently, $B_{2n+1,m}$ admit Elegant Fuzzy Labeling and (μ, ρ) is an Elegant Fuzzy Labeling of $B_{2n+1,m}$. □

Example 3.4. To illustrate Theorem 3.3, we consider a bistar graph $B_{3,4}$ in Fig.3. By the theorem, an Elegant Fuzzy Labeling (μ, ρ) of $B_{3,4}$ is given as follows: $\mu(v_0) = 0.69$, $\mu(v_1) = 1$, $\mu(v_2) = 0.54$, $\mu(u_0) = 0.62$, $\mu(u_1) = 0.77$. $\mu(u_2) = 0.85$, $\mu(u_3) = 0.92$, $\rho(v_0v_1) = 0.07$, $\rho(v_0v_2) = 0.15$, $\rho(v_0u_0) = 0.23$, $\rho(u_0u_1) = 0.31$, $\rho(u_0u_2) = 0.38$, $\rho(u_0u_3) = 0.46$. From this, we observe that all edge labels are distinct, nonzero, and strictly less than the labels of their respective endpoints. Hence, $B_{3,4}$ admits Elegant Fuzzy Labeling, as established by Theorem 3.3.

Theorem 3.5. *The graphs $S_m + S_n$ admit Elegant Fuzzy Labeling for every m, n.*

Proof. Consider $S_m + S_n$ with $m + n$ vertices, $mn + m + n - 2$ edges, with vertex set $V(S_m + S_n) = \{v_0, v_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{n-1}\}$ and edge set $E(S_m + S_n) = \{v_iu_j; 0 \leq i \leq m-1, 0 \leq j \leq n-1, v_0v_i, u_0u_j; 1 \leq i \leq m-1, 1 \leq j \leq n-1\}$. Let $f : V(S_m + S_n) \to \{mn + m + n - 1, mn + m + n, \ldots, 2(mn + m + n) - 3\}$ be an injective function defined as,

$$f(v_i) = \begin{cases} mn + m + n - 1 & i = 0 \\ 2mn + 2m + 2n - (i + 2) & 1 \leq i \leq m - 1, \end{cases}$$

$$f(u_j) = \begin{cases} 2mn + m + 2n - 2 & j = 0 \\ j(m + 1) + mn + m + n - 1 & 1 \leq j \leq n - 1. \end{cases}$$

Let g be a function from $E(S_m + S_n)$ into the set $\{0, 1, \ldots, mn+m+n-2\}$ defined as,
$g(v_iu_j) = (f(v_i) + f(u_j)) \bmod (mn + m + n - 1)$ $0 \leq i \leq m - 1,\ 0 \leq j \leq n - 1$,
$g(v_0v_i) = (f(v_0) + f(v_i)) \bmod (mn + m + n - 1),\ 1 \leq i \leq m - 1$,
$g(u_0u_j) = (f(u_0) + f(u_j)) \bmod (mn + m + n - 1),\ 1 \leq j \leq n - 1$.

Now, for $1 \leq i \leq m - 1$,

$$\begin{aligned} g(v_0v_i) &= (f(v_0) + f(v_i)) \bmod (mn + m + n - 1) \\ &= (mn + m + n - 1 + 2mn + 2m + 2n - (i + 2)) \bmod (mn + m + n - 1) \\ &= (2(mn + m + n - 1) + mn + m + n - 1 - i) \bmod (mn + m + n - 1) \\ &= mn + m + n - 1 - i. \end{aligned}$$

For $1 \leq j \leq n-1$,

$$\begin{aligned}
g(u_0u_j) &= (f(u_0) + f(u_j)) \bmod (mn+m+n-1) \\
&= (2mn+m+2n-2+jm+j+mn+m+n-1) \bmod (mn+m+n-1) \\
&= (3(mn+m+n-1)+(j-1)m+j) \bmod (mn+m+n-1) \\
&= (j-1)m+j.
\end{aligned}$$

For $i = 0, j = 0$,

$$\begin{aligned}
g(v_0u_0) &= (f(v_0) + f(u_0)) \bmod (mn+m+n-1) \\
&= (mn+m+n-1+2mn+m+2n-2) \bmod (mn+m+n-1) \\
&= (2(mn+m+n-1)+mn+n-1) \bmod (mn+m+n-1) \\
&= mn+n-1.
\end{aligned}$$

For $i = 0, 1 \leq j \leq n-1$,

$$\begin{aligned}
g(v_0u_j) &= (f(v_0) + f(u_j)) \bmod (mn+m+n-1) \\
&= (mn+m+n-1+jm+j+mn+m+n-1) \bmod (mn+m+n-1) \\
&= (2(mn+m+n-1)+jm+j) \bmod (mn+m+n-1) \\
&= jm+j.
\end{aligned}$$

For $1 \leq i \leq m-1, j = 0$,

$$\begin{aligned}
g(v_iu_0) &= (f(v_i) + f(u_0)) \bmod (mn+m+n-1) \\
&= (2mn+2m+2n-i-2+2mn+m+2n-2) \bmod (mn+m+n-1) \\
&= (3(mn+m+n-1)+mn+n-i-1) \bmod (mn+m+n-1) \\
&= mn+n-i-1.
\end{aligned}$$

For $1 \leq i \leq m-1, 1 \leq j \leq n-1$,

$$\begin{aligned}
g(v_iu_j) &= (f(v_i) + f(u_j)) \bmod (mn+m+n-1) \\
&= (2mn+2m+2n-i-2+jm+j+mn+ \\
&\qquad +m+n-1) \bmod (mn+m+n-1) \\
&= (3(mn+m+n-1)+jm+j-i) \bmod (mn+m+n-1) \\
&= jm+j-i.
\end{aligned}$$

Hence,

$$g(v_0v_i) = mn + m + n - 1 - i,\ 1 \le i \le m-1,$$

$$g(u_0u_j) = (j-1)m + j,\ 1 \le j \le n-1,$$

$$g(v_iu_j) = \begin{cases} mn+n-1 & i=0, j=0 \\ jm+j & i=0, 1\le j \le n-1 \\ mn+n-i-1 & 1 \le i \le m-1, j=0 \\ jm+j-i & 1 \le i \le m-1, 1 \le j \le n-1. \end{cases}$$

Now, we have $h = \frac{1}{2(mn+m+n)-3}$. The vertex labeling $\mu : V(S_m + S_n) \to [0,1]$ is defined as, $\mu(v_i) = f(v_i).\frac{1}{2(mn+m+n)-3}$ for every $0 \le i \le m-1$ and $\mu(u_j) = f(u_j).\frac{1}{2(mn+m+n)-3}$ for every $0 \le j \le n-1$ and edge labeling $\rho : E(S_m + S_n) \to [0,1]$ is defined as, $\rho(v_iu_j) = g(v_iu_j).\frac{1}{2(mn+m+n)-3}$ for every $0 \le i \le m-1,\ 0 \le j \le n-1,\ \rho(v_0v_i) = g(v_0v_i).\frac{1}{2(mn+m+n)-3}$ for every $1 \le i \le m-1,\ \rho(u_0u_j) = g(u_0u_j).\frac{1}{2(mn+m+n)-3}$ for every $1 \le j \le n-1$.

It can be verified that the sets $\{g(v_0v_i),\ 1 \le i \le m-1\}, \{g(u_0u_j),\ 1 \le j \le n-1\}$ and $\{g(v_iu_j),\ \ 0 \le i \le m-1, 0 \le j \le n-1\}$ are mutually disjoint and the elements are distinct and nonzero, therefore the edge labels are distinct and nonzero. Further $\max\{(g(v_0v_i), g(u_0u_j), g(v_iu_j)\} = mn+m+n-2$ and $\min\{(f(v_i), f(u_j))\} = mn+m+n-1$. Therefore, $g(v_iu_j) < f(v_i) \wedge f(u_j)$ for every $0 \le i \le m-1,\ 0 \le j \le n-1, g(v_0v_i) < f(v_0) \wedge f(v_i)$ for every $1 \le i \le m-1$ and $g(u_0u_j) < f(u_0) \wedge f(u_j)$ for every $1 \le j \le n-1$. Hence, the edge labels are less than the minimum of their respective endpoints' labels. Consequently, $S_m + S_n$ admit Elegant Fuzzy Labeling and (μ, ρ) is an Elegant Fuzzy Labeling of $S_m + S_n$. □

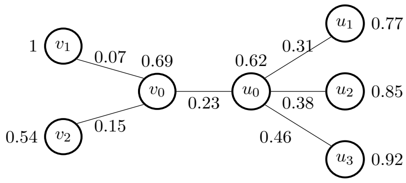

Figure 3. $B_{3,4}$

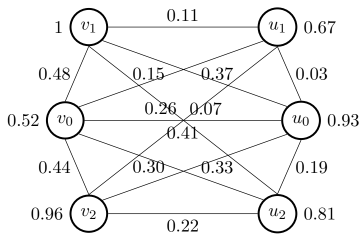

Figure 4. $S_3 + S_3$

Example 3.6. To illustrate Theorem 3.5, we consider a graph $S_3 + S_3$ in Fig.4. By the theorem, an Elegant Fuzzy Labeling (μ, ρ) of $S_3 + S_3$ is given as follows: $\mu(v_0) = 0.52,\ \mu(v_1) = 1,\ \mu(v_2) = 0.96,\ \mu(u_0) = 0.93,\ \mu(u_1) = 0.67.\ \mu(u_2) = 0.81,\ \rho(v_0u_0) = 0.41,\ \rho(v_0v_1) = 0.48,\ \rho(v_0v_2) = 0.44,\ \rho(u_0u_1) = 0.03,\ \rho(u_0u_2) = 0.19,\ \rho(v_0u_1) = 0.15,\ \rho(v_0u_2) = 0.30,\ \rho(v_1u_0) = 0.37,\ \rho(v_1u_1) = 0.11,\ \rho(v_1u_2) =$

$0.26, \rho(v_2u_0) = 0.33,\ \rho(v_2u_1) = 0.07,\ \rho(v_2u_2) = 0.22$. From this, we observe that all edge labels are distinct, nonzero, and strictly less than the labels of their respective endpoints. Hence, S_3+S_3 admits Elegant Fuzzy Labeling, as established by Theorem 3.5.

4. Discussion

This section explores Elegant Fuzzy Labeling in comparison with traditional fuzzy labeling and examines its boundary conditions.

Elegant Fuzzy Labeling differs from traditional labeling methods by introducing a "fuzzy" element to elegant labeling, introduced by Chang et al. [2], which allows for handling uncertainty and impreciseness in a more effective way. Elegant Fuzzy Labeling, while building on the foundation of traditional fuzzy graph theory, introduces stricter conditions to ensure a more precise and structured labeling process. Our previous work [10] established that a graph admits Elegant Fuzzy Labeling if and only if it admits elegant labeling and that every graph admitting Elegant Fuzzy Labeling necessarily admits fuzzy labeling, but not vice versa. These results highlight that Elegant Fuzzy Labeling is a constrained subset of fuzzy labeling, offering a stricter yet more structured approach.

In Elegant Fuzzy Labeling, the parameters h' and h define its boundary conditions. The translation constant h' is a positive constant that can be any suitable value depending on the context of the labeling. It is generally chosen to ensure that the labeling process remains flexible and adaptable to different graph structures. Thus, the boundary for h' is not fixed and can extend infinitely, providing flexibility in the application of Elegant Fuzzy Labeling to various graph structures and scenarios. On the other hand, h is bounded by 0 and $\frac{1}{h'+1}$. Therefore, h' and h work together to control the degree of fuzziness. As the value of h' increases (which corresponds to a larger degree of fuzziness or uncertainty), h decreases to ensure that the labeling process does not become too vague and remains manageable.

5. Conclusion

In the present paper, we have extended the concept of Elegant Fuzzy Labeling to star graphs and their variants. By maintaining the same translation constant and contraction scale, we have proved that star graphs S_n, bistar graphs $B_{2n+1,m}$

and the join of two star graphs $S_m + S_n$, admit Elegant Fuzzy Labeling. By providing a structured approach to labeling graphs under fuzzy constraints, our results contribute to both theoretical graph studies and practical implementations.

Future research directions include extending Elegant Fuzzy Labeling to other graph families, such as trees, grid graphs, and hypergraphs, to explore its adaptability across different topologies. Additionally, we can explore the possibility of extending Elegant Fuzzy Labeling to soft settings, where handling uncertainty and imprecise information may offer new insights. Furthermore, future studies can investigate potential applications of Elegant Fuzzy Labeling in various domains.

DECLARATIONS

Conflicts of Interest: None of the authors have any conflict of interest.

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