

GENERALIZED FRACTIONAL HERMITE HADAMARD TYPE INEQUALITIES FOR (s, m) -CONVEX FUNCTIONS IN THE 1st AND 2nd KINDS

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ABSTRACT. Hermite-Hadamard type inequalities are among the most well-known integral inequalities in mathematics. In this article, we have established several results concerning fractional Hermite-Hadamard type inequalities for twice differentiable (s, m) -convex functions of both the 1st and 2nd kinds. These findings offer a broader generalization compared to the results presented in [9].

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on an interval I of real numbers, and let $a, b \in I$ with $a < b$. Then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2},$$

is known as the Hermite–Hadamard inequality for convex functions [27]. Both inequalities hold in the reverse direction when f is concave. This inequality may be viewed as a refinement of convexity and follows directly from Jensen’s inequality.

The study of convexity and its extensions plays a fundamental role in mathematical analysis, optimization theory, and applied sciences. Among the many inequalities related to convex functions, the Hermite–Hadamard inequality occupies a central place due to its wide applications in numerical analysis, probability theory, approximation theory, and fractional calculus. This classical result, which provides

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bounds for the average value of a convex function in terms of its endpoints, has been extensively investigated and generalized in various directions (see [5, 7, 11, 18, 24]).

In recent years, fractional calculus has emerged as a powerful branch of analysis, extending classical ideas through differentiation and integration of non-integer order. The integration of fractional operators into Hermite–Hadamard type inequalities has produced numerous generalizations that yield sharper estimates and deeper insights into the behavior of convex functions in fractional settings (see [2, 15, 16, 20, 26]).

At the same time, several generalizations of convexity, such as s -convexity, m -convexity, and their hybrid extensions have been introduced to study wider classes of functions that are not strictly convex but still retain convex-like properties. Within this framework, the class of (s, m) -convex functions of the first and second kinds has attracted particular interest due to its flexibility and potential for diverse applications (see [4, 17, 29]).

Motivated by these advances, this paper establishes new generalized fractional Hermite–Hadamard type inequalities for (s, m) -convex functions of both the first and second kinds. By employing fractional integral operators, we derive results that extend, unify, and complement existing inequalities in the literature. Our contributions not only enhance the theoretical framework of convexity and fractional calculus but also provide useful tools for mathematical modeling, numerical methods, and related applications (see [1, 12, 13, 23]).

In this context, let's revisit the definitions of s -convex functions of the 1st and 2nd kinds, as outlined in [8].

Definition 1.1. A function $f : I \rightarrow \mathbb{R}$ is said to be s -convex in the 1st kind, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t^s)f(y),$$

$\forall x, y \in I, t \in [0, 1]$ and $s \in (0, 1]$.

Definition 1.2. A function $f : I \rightarrow \mathbb{R}$ is said to be s -convex in the 2nd kind, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

$\forall x, y \in I, t \in [0, 1]$ and $s \in (0, 1]$.

In 1984, Toader defined the following class of m -convex functions in [28] as:

Definition 1.3. A function $f : [0, b] \rightarrow \mathbb{R}$ is said to be m -convex, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

$\forall x, y \in [0, b], t \in [0, 1]$ and $m \in (0, 1]$. It may be observed that if we put $m = 1$ in the above definition then m -convexity changes into normal convexity (see [25]).

In 1993, Mihešan defined the class of (s, m) -convex functions of the 1st kind by unifying the concepts of m -convexity and 1st kind s -convexity (see [19]):

Definition 1.4. A function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (s, m) -convex in the 1st kind, if we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s)f(y)$$

$\forall x, y \in [0, b], t \in [0, 1]$ and $s, m \in (0, 1]$.

If we put $m = 1$ and $s = 1$ one by one, then the above definition becomes the definition of s -convex function in the 1st kind and m -convex function respectively. Also if we put both $m = s = 1$ simultaneously then we get normal convexity (see [25]).

In 2014, Eftekhari proposed the class of (s, m) -convex functions of the 2nd kind by merging the definitions of m -convex functions and s -convex functions of the 2nd kind (see [13]):

Definition 1.5. A function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (s, m) -convex in the 2nd kind, if we have

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in [0, b], t \in [0, 1]$ and $s, m \in (0, 1]$.

If we put $m = 1$ and $s = 1$ one by one, then the above definition becomes the definition of s -convex function in the 2nd kind and m -convex function respectively. Also if we put both $m = s = 1$ simultaneously then we get normal convexity (see [25]).

Theorem 1.6. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous and non-negative functions and let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ then,

$$\int_a^b |f(x)g(x)|dx = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx \right)^{\frac{1}{q}}.$$

The above inequality is known as Hölder's inequality (see [6]). Note that if we put $p = q = 2$, the above inequality becomes Cauchy – Schwarz inequality and if we put $q = 1$ so $p = \infty$, then $\|f\|_\infty$ stands for the essential supremum of $|f|$ or vice versa.

Theorem 1.7. Let f, g be real-valued functions defined on $[a, b]$ and if $|f|, |f||g|^q$ are integrable on $[a, b]$, then for $q \geq 1$ we have,

$$\int_a^b |f(x)||g(x)|dx \leq \left(\int_a^b |f(x)|dx \right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)||g(x)|^q dx \right)^{\frac{1}{q}}.$$

The above inequality is well known as power mean inequality (see [21]).

We will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1.8. [27] Let $f \in L[a, b]$. The Riemann – Liouville integrals $J_{a+}^\alpha f(x)$ and $J_{b-}^\alpha f(x)$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

We now present some previously established results on the fractional Hermite Hadamard type inequalities for twice differentiable m -convex functions, as captured in [9].

Theorem 1.9. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and $f'' \in L[a, b]$. If $|f''|$ is m -convex on I for some $m \in (0, 1]$, then the following inequality involving fractional integrals holds for $\alpha > 0$:

$$(1) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \leq \frac{\alpha(b-a)^2}{6(\alpha+1)(\alpha+3)} \\ \times \left[|f''(a)| + |f''(b)| + m \frac{(\alpha+5)}{2(\alpha+2)} \left(\left| f''\left(\frac{a}{m}\right) \right| + \left| f''\left(\frac{b}{m}\right) \right| \right) \right].$$

Theorem 1.10. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and $f'' \in L[a, b]$. If $|f''|^q$ is m -convex on I for some $m \in (0, 1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds:*

$$(2) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \frac{(b - a)^2}{2(\alpha + 1)} \mathbf{B}^{\frac{1}{p}}(p + 1, \alpha p + 1) \times \left[\left(\frac{|f''(a)|^q + m \left| f''\left(\frac{b}{m}\right) \right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{m \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right],$$

where \mathbf{B} is the Euler Beta function.

Theorem 1.11. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and $f'' \in L[a, b]$. If $|f''|^q$ is m -convex on I for some $m \in (0, 1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$(3) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \frac{\alpha(b - a)^2}{4(\alpha + 1)(\alpha + 2)(3(\alpha + 3))^{\frac{1}{q}}} \left[\left(2(\alpha + 2)|f''(a)|^q + m(\alpha + 5) \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(m(\alpha + 5) \left| f''\left(\frac{a}{m}\right) \right|^q + 2(\alpha + 2)|f''(b)|^q \right)^{\frac{1}{q}} \right].$$

The structure of this article unfolds as follows: In the subsequent sections, we aim to establish six unique outcomes concerning fractional Hermite Hadamard type inequalities applicable to the category of twice differentiable (s, m) -convex functions of 1st and 2nd kinds. Our approach will leverage diverse techniques, encompassing Hölder’s and power mean integral inequalities. These findings are anticipated to exhibit a broader scope compared to those presented in [3, 8, 9] and [22]. The fourth section will provide a concluding statement, while the final section will offer insights and future prospects for readers interested in further exploration.

In order to prove our main theorems we need the following lemma from [3].

Lemma 1.12. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and $f'' \in L[a, b]$. Then,*

for any $\alpha > 0$, the following identity involving fractional integrals holds:

$$(4) \quad \begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\ &= \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 t(1-t^\alpha) [f''(ta + (1-t)b) + f''((1-t)a + tb)] dt, \end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

2. VARIOUS ESTIMATIONS OF RIGHT BOUND OF FRACTIONAL HERMITE HADAMARD TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE (s, m)-CONVEX FUNCTIONS OF THE 1st KIND

We begin by presenting our results concerning twice differentiable (s, m)-convex functions of the 1st kind.

Theorem 2.1. *Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and that $f'' \in L[a, b]$. If $|f''|$ is (s, m)-convex of the 1st kind on I for some $s, m \in (0, 1]$, then the following inequality involving fractional integrals holds for any $\alpha > 0$:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\alpha(b-a)^2}{2(\alpha+1)} \left[\frac{1}{(s+2)(\alpha+s+2)} (|f''(a)| + |f''(b)|) + \right. \\ & \quad \left. m \left(\frac{1}{2(\alpha+2)} - \frac{1}{(s+2)(\alpha+s+2)} \right) \left(\left| f''\left(\frac{a}{m}\right) \right| + \left| f''\left(\frac{b}{m}\right) \right| \right) \right]. \end{aligned}$$

Proof. By applying Lemma 1.12 and the properties of absolute value, we obtain

$$(5) \quad \begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t(1-t^\alpha)| [|f''(ta + (1-t)b)| + |f''((1-t)a + tb)|] dt. \end{aligned}$$

Since $|f''|$ is an (s, m)-convex function of the 1st kind, we can use

$$|f''(ta + (1-t)b)| \leq t^s |f''(a)| + m(1-t^s) \left| f''\left(\frac{b}{m}\right) \right|$$

and

$$|f''((1-t)a + tb)| \leq m(1-t^s) \left| f''\left(\frac{a}{m}\right) \right| + t^s |f''(b)|.$$

Using these results, inequality (5) becomes:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{2(\alpha + 1)} \int_0^1 \left[t^{s+1}(1-t^\alpha) |f''(a)| + mt(1-t^\alpha)(1-t^s) \left| f''\left(\frac{b}{m}\right) \right| \right. \\ & \quad \left. + mt(1-t^\alpha)(1-t^s) \left| f''\left(\frac{a}{m}\right) \right| + t^{s+1}(1-t^\alpha) |f''(b)| \right] dt. \end{aligned}$$

After rearranging the terms and using the following integrals, the result in Theorem 2.1 is obtained.

$$\int_0^1 t^{s+1}(1-t^\alpha) dt = \frac{\alpha}{(s+2)(\alpha+s+2)}$$

and

$$\int_0^1 t(1-t^\alpha)(1-t^s) dt = \frac{\alpha}{2(\alpha+2)} - \frac{\alpha}{(s+2)(\alpha+s+2)}.$$

□

Remark 2.2. The following well-established results can be obtained as special cases of our derived result by varying the values of s , m , and α :

- (1) By setting $m = 1$ or $s = m = 1$ or $s \rightarrow 0$ or $s \rightarrow 0$ with $m = 1$ in Theorem 2.1, we obtain the result presented in Theorem 2 of [3].
- (2) By setting $s = 1$ in Theorem 2.1, we obtain the result presented in Theorem 1.9.
- (3) By setting $\alpha = s = m = 1$ or $\alpha = m = 1$ or $s \rightarrow 0$ with $\alpha = 1$ in Theorem 2.1, we obtain the result presented in Remark 1 of [22].
- (4) By setting $\alpha = s = 1$ in Theorem 2.1, we obtain the result presented in Corollary 1 of [9].
- (5) By setting $\alpha = m = 1$ with $s \rightarrow 0$ in Theorem 2.1, we obtain the second result presented in Theorem 3 of [8].

Corollary 2.3. *If we choose $\alpha = 1$ in Theorem 2.1, then we get the following Hermite Hadamard type inequality for twice differentiable (s, m) -convex function of*

1st kind:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(\zeta) d\zeta \right| \leq \frac{(b-a)^2}{24(s+2)(s+3)} \times \\ \left[6(|f''(a)| + |f''(b)|) + ms(s+5) \left(\left| f''\left(\frac{a}{m}\right) \right| + \left| f''\left(\frac{b}{m}\right) \right| \right) \right].$$

Theorem 2.4. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and that $f'' \in L[a, b]$. If $|f''|^q$ is (s, m) -convex of the 1st kind on I for some $s, m \in (0, 1]$ and $q \geq 1$, then the following inequality involving fractional integrals holds for any $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ \leq \frac{(b-a)^2}{2(\alpha+1)(s+1)^{\frac{1}{q}}} \mathbf{B}^{\frac{1}{p}}(p+1, \alpha p+1) \left[\left(|f''(a)|^q + ms \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left(ms \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right],$$

where \mathbf{B} is Euler Beta function.

Proof. By applying Lemma 1.12 and the properties of absolute value, we obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ (6) \quad \leq \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 |t(1-t^\alpha)| [|f''(ta + (1-t)b)| + |f''((1-t)a + tb)|] dt.$$

Applying Definition 1.6 to $\int_0^1 |t(1-t^\alpha)| |f''(ta + (1-t)b)| dt$ and $\int_0^1 |t(1-t^\alpha)| |f''((1-t)a + tb)| dt$ implies

$$\int_0^1 |t(1-t^\alpha)| |f''(ta + (1-t)b)| dt \\ \leq \left(\int_0^1 |t(1-t^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}$$

and

$$\int_0^1 |t(1 - t^\alpha)| |f''((1 - t)a + tb)| dt \leq \left(\int_0^1 |t(1 - t^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''((1 - t)a + tb)|^q dt \right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is an (s, m) -convex function of the 1st kind, we can use

$$|f''(ta + (1 - t)b)|^q \leq t^s |f''(a)|^q + m(1 - t^s) \left| f''\left(\frac{b}{m}\right) \right|^q$$

and

$$|f''((1 - t)a + tb)|^q \leq m(1 - t^s) \left| f''\left(\frac{a}{m}\right) \right|^q + t^s |f''(b)|^q.$$

Using these results, inequality (6) becomes:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b - a)^2}{2(\alpha + 1)} \left(\int_0^1 t^p (1 - t^\alpha)^p dt \right)^{\frac{1}{p}} \\ & \times \left[\left(\int_0^1 \left[t^s |f''(a)|^q + m(1 - t^s) \left| f''\left(\frac{b}{m}\right) \right|^q \right] dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 \left[m(1 - t^s) \left| f''\left(\frac{b}{m}\right) \right|^q + t^s |f''(b)|^q \right] dt \right)^{\frac{1}{q}} \right]. \end{aligned}$$

After rearranging the terms and using the following integrals, the result in Theorem 2.4 is obtained.

$$\int_0^1 t^s dt = \frac{1}{s + 1}, \quad \int_0^1 (1 - t^s) dt = \frac{s}{s + 1}$$

and

$$\int_0^1 t^p (1 - t^\alpha)^p dt \leq \int_0^1 t^p (1 - t)^{\alpha p} dt = \mathbf{B}(p + 1, \alpha p + 1).$$

□

Remark 2.5. The following well-established results can be obtained as special cases of our derived result by varying the values of s , m , and α :

- (1) By setting $s = 1$ in Theorem 2.4, we obtain the result presented in Theorem 1.10.
- (2) By setting $m = 1$ in Theorem 2.4, we obtain the second result presented in Corollary 3.6 of [10].
- (3) By setting $s = m = 1$ in Theorem 2.4, we obtain the result presented in Theorem 3 of [3].
- (4) By setting $\alpha = m = 1$ in Theorem 2.4, we obtain the third result presented in Corollary 3.6 of [10].
- (5) By setting $\alpha = s = 1$ in Theorem 2.4, we obtain the third result presented in Corollary 2 of [9].
- (6) By setting $\alpha = s = m = 1$ in Theorem 2.4, we obtain the result presented in Remark 13 of [14].
- (7) By setting $s \rightarrow 0$ with $\alpha = m = 1$ in Theorem 2.4, we obtain the second result presented in Theorem 4 of [8].

Corollary 2.6. *Under the assumptions of the Theorem 2.4,*

- (1) *If we choose $p = q = 2$, then we get the result attained by using Cauchy–Schwarz integral inequality as:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{2(\alpha+1)(s+1)^{\frac{1}{2}}} \mathbf{B}^{\frac{1}{2}}(3, 2\alpha+1) \left[\left(|f''(a)|^2 + ms \left| f''\left(\frac{b}{m}\right) \right|^2 \right)^{\frac{1}{2}} \right. \\ & \quad \left. + \left(ms \left| f''\left(\frac{a}{m}\right) \right|^2 + |f''(b)|^2 \right)^{\frac{1}{2}} \right]. \end{aligned}$$

- (2) *If we choose $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable (s, m) -convex function of 1st kind:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(\zeta) d\zeta \right| \leq \frac{(b-a)^2}{4(s+1)^{\frac{1}{q}}} \mathbf{B}^{\frac{1}{p}}(p+1, p+1) \times \\ & \left[\left(|f''(a)|^q + ms \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(ms \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

(3) If we choose $s \rightarrow 0$ or $s \rightarrow 0$ with $m = 1$, then we get the following Fractional Hermite Hadamard type inequality for twice differentiable Quasi-convex function:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b - a)^2}{2(\alpha + 1)} \mathbf{B}^{\frac{1}{p}}(p + 1, \alpha p + 1) (|f''(a)| + |f''(b)|). \end{aligned}$$

(4) If we choose $s \rightarrow 0$ with $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable Quasi- m -convex function:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(\zeta) d\zeta \right| \\ & \leq \frac{(b - a)^2}{4} \mathbf{B}^{\frac{1}{p}}(p + 1, p + 1) (|f''(a)| + |f''(b)|). \end{aligned}$$

Theorem 2.7. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and that $f'' \in L[a, b]$. If $|f''|^q$ is (s, m) -convex of the 1st kind on I for some $s, m \in (0, 1]$ and $q \geq 1$, then the following inequality involving fractional integrals holds for any $\alpha > 0$:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\alpha(b - a)^2}{2(\alpha + 1)} \left(\frac{1}{2(\alpha + 2)} \right)^{1 - \frac{1}{q}} \left[\left(\frac{1}{(s + 2)(\alpha + s + 2)} |f''(a)|^q + \right. \right. \\ & \quad \left. \left. m \left(\frac{1}{2(\alpha + 2)} - \frac{1}{(s + 2)(\alpha + s + 2)} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(m \left(\frac{1}{2(\alpha + 2)} - \frac{1}{(s + 2)(\alpha + s + 2)} \right) \left| f'' \left(\frac{a}{m} \right) \right|^q + \right. \right. \\ & \quad \left. \left. \frac{1}{(s + 2)(\alpha + s + 2)} |f''(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. By applying Lemma 1.12 and the properties of absolute value, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ (7) \quad & \leq \frac{(b - a)^2}{2(\alpha + 1)} \int_0^1 |t(1 - t^\alpha)| [|f''(ta + (1 - t)b)| + |f''((1 - t)a + tb)|] dt. \end{aligned}$$

Applying Definition 1.7 to $\int_0^1 |t(1-t^\alpha)| |f''(ta+(1-t)b)| dt$ and $\int_0^1 |t(1-t^\alpha)| |f''((1-t)a+tb)| dt$ implies

$$\begin{aligned} & \int_0^1 |t(1-t^\alpha)| |f''(ta+(1-t)b)| dt \\ & \leq \left(\int_0^1 t(1-t^\alpha) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t(1-t^\alpha) |f''(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |t(1-t^\alpha)| |f''((1-t)a+tb)| dt \\ & \leq \left(\int_0^1 t(1-t^\alpha) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t(1-t^\alpha) |f''((1-t)a+tb)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f''|^q$ is an (s, m) -convex function of the 1^{st} kind, we can use

$$|f''(ta+(1-t)b)|^q \leq t^s |f''(a)|^q + m(1-t^s) \left| f''\left(\frac{b}{m}\right) \right|^q$$

and

$$|f''((1-t)a+tb)|^q \leq m(1-t^s) \left| f''\left(\frac{a}{m}\right) \right|^q + t^s |f''(b)|^q.$$

Using these results, inequality (7) becomes:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{2(\alpha+1)} \left(\int_0^1 t(1-t^\alpha) dt \right)^{1-\frac{1}{q}} \times \\ & \left[\left(|f''(a)|^q \int_0^1 t^{s+1}(1-t^\alpha) dt + m \left| f''\left(\frac{b}{m}\right) \right|^q \int_0^1 t(1-t^s)(1-t^\alpha) dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(m \left| f''\left(\frac{a}{m}\right) \right|^q \int_0^1 t(1-t^s)(1-t^\alpha) dt + |f''(b)|^q \int_0^1 t^{s+1}(1-t^\alpha) dt \right)^{\frac{1}{q}} \right]. \end{aligned}$$

After rearranging the terms and using the following integrals, the result in Theorem 2.1 is obtained.

$$\int_0^1 t(1-t^\alpha)dt = \frac{\alpha}{2(\alpha+2)}, \quad \int_0^1 t^{s+1}(1-t^\alpha)dt = \frac{\alpha}{(s+2)(\alpha+s+2)}$$

and

$$\int_0^1 t(1-t^s)(1-t^\alpha)dt = \frac{\alpha}{2(\alpha+2)} - \frac{\alpha}{(s+2)(\alpha+s+2)}.$$

□

Remark 2.8. The following well-established results can be obtained as special cases of our derived result by varying the values of s , m , and α :

- (1) By setting $s = 1$ in Theorem 2.7, we obtain the result presented in Theorem 1.11.
- (2) By setting $m = 1$ in Theorem 2.7, we obtain the second result presented in Corollary 3.9 of [10].
- (3) By setting $s = m = 1$ in Theorem 2.7, we obtain the result presented in Theorem 4 of [3].
- (4) By setting $\alpha = m = 1$ in Theorem 2.7, we obtain the third result presented in Corollary 3.9 of [10].
- (5) By setting $\alpha = s = 1$ in Theorem 2.7, we obtain the result presented in Corollary 3 of [9].
- (6) By setting $\alpha = s = m = 1$ in Theorem 2.7, we obtain the third result presented in Corollary 3 of [8].
- (7) By setting $s \rightarrow 0$ with $\alpha = m = 1$ in Theorem 2.7, we obtain the second result presented in Theorem 5 of [8].
- (8) By setting $s \rightarrow 0$ with $\alpha = 1$ in Theorem 2.7, we obtain the result presented in Remark 1 of [22].
- (9) By setting $s \rightarrow 0$ or $s \rightarrow 0$ with $m = 1$ in Theorem 2.7, we obtain the result presented in Theorem 2 of [3].

Corollary 2.9. *If we choose $\alpha = 1$ in Theorem 2.7, then we get the following Hermite Hadamard type inequality for twice differentiable (s, m) -convex function of*

1st kind:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(\zeta) d\zeta \right| \leq \frac{(b-a)^2}{24\{(s+2)(s+3)\}^{\frac{1}{q}}} \times \left[\left(6|f''(a)|^q + ms(s+5) \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(ms(s+5) \left| f''\left(\frac{a}{m}\right) \right|^q + 6|f''(b)|^q \right)^{\frac{1}{q}} \right].$$

3. VARIOUS ESTIMATIONS OF RIGHT BOUND OF FRACTIONAL HERMITE HADAMARD TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE (s, m) -CONVEX FUNCTIONS OF THE 2nd KIND

We will now present several results related to (s, m) -convex functions of the 2nd kind.

Theorem 3.1. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and that $f'' \in L[a, b]$. If $|f''|$ is (s, m) -convex of the 2nd kind on I for some $s, m \in (0, 1]$, then the following inequality involving fractional integrals holds for any $\alpha > 0$:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\alpha(b-a)^2}{2(\alpha+1)} \left[\frac{1}{(s+2)(\alpha+s+2)} (|f''(a)| + |f''(b)|) + \right. \\ & \left. m \left(\frac{\mathbf{B}(2, s+1) - \mathbf{B}(\alpha+2, s+1)}{\alpha} \right) \left(\left| f''\left(\frac{a}{m}\right) \right| + \left| f''\left(\frac{b}{m}\right) \right| \right) \right]. \end{aligned}$$

Proof. The above outcome can be demonstrated through a method analogous to that used in the proof of Theorem 2.1, relying on the definition of (s, m) -convex functions of the 2nd kind. \square

Remark 3.2. The following well-established results can be obtained as special cases of our derived result by varying the values of s , m , and α :

- (1) By setting $m = 1$ in Theorem 3.1, we obtain the result presented in Theorem 2.2 of [27].
- (2) By setting $s = m = 1$ in Theorem 3.1, we obtain the result presented in Theorem 2 of [3].
- (3) By setting $s = 1$ in Theorem 3.1, we obtain the result presented in Theorem 1.9.

- (4) By setting $\alpha = s = m = 1$ in Theorem 3.1, we obtain the result presented in Remark 1 of [22].
- (5) By setting $\alpha = s = 1$ in Theorem 3.1, we obtain the result presented in Corollary 1 of [9].
- (6) By setting $\alpha = m = 1$ in Theorem 3.1, we obtain the second result presented in Corollary 1 of [8].
- (7) By setting $\alpha = m = 1$ with $s \rightarrow 0$ in Theorem 2.1, we obtain the third result presented in Corollary 1 of [8].

Corollary 3.3. *Under the assumptions of the Theorem 3.1,*

- (1) *If we choose $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable (s, m) -convex function of 2nd kind:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(\zeta) d\zeta \right| \\ & \leq \frac{(b-a)^2}{4(s+2)(s+3)} \left[|f''(a)| + |f''(b)| + m \left(\left| f''\left(\frac{a}{m}\right) \right| + \left| f''\left(\frac{b}{m}\right) \right| \right) \right]. \end{aligned}$$

- (2) *If we choose $s \rightarrow 0$, then we get the following Fractional Hermite Hadamard type inequality for twice differentiable $P - m$ -convex function:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\alpha(b-a)^2}{4(\alpha + 1)(\alpha + 2)} \left[(|f''(a)| + |f''(b)|) + m \left(\left| f''\left(\frac{a}{m}\right) \right| + \left| f''\left(\frac{b}{m}\right) \right| \right) \right]. \end{aligned}$$

- (3) *If we choose $s \rightarrow 0$ with $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable $P - m$ -convex function:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(\zeta) d\zeta \right| \\ & \leq \frac{(b-a)^2}{24} \left[(|f''(a)| + |f''(b)|) + m \left(\left| f''\left(\frac{a}{m}\right) \right| + \left| f''\left(\frac{b}{m}\right) \right| \right) \right]. \end{aligned}$$

- (4) *If we choose $s \rightarrow 0$ with $m = 1$, then we get the following Fractional Hermite Hadamard type inequality for twice differentiable P -convex function:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\alpha(b-a)^2}{2(\alpha + 1)(\alpha + 2)} (|f''(a)| + |f''(b)|). \end{aligned}$$

Theorem 3.4. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and that $f'' \in L[a, b]$. If $|f''|^q$ is (s, m) -convex of the 2nd kind on I for some $s, m \in (0, 1]$ and $q \geq 1$, then the following inequality involving fractional integrals holds for any $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{2(\alpha+1)(s+1)^{\frac{1}{q}}} \mathbf{B}^{\frac{1}{p}}(p+1, \alpha p+1) \left(\left[|f''(a)|^q + m \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. \left[m \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right]^{\frac{1}{q}} \right), \end{aligned}$$

where \mathbf{B} is Euler Beta function.

Proof. The above outcome can be demonstrated through a method analogous to that used in the proof of Theorem 2.4, relying on the definition of (s, m) -convex functions of the 2nd kind. \square

Remark 3.5. The following well-established results can be obtained as special cases of our derived result by varying the values of s , m , and α :

- (1) By setting $s = 1$ in Theorem 3.4, we obtain the result presented in Theorem 1.10.
- (2) By setting $m = 1$ in Theorem 3.4, we obtain the result presented in Theorem 2.4 of [27].
- (3) By setting $s = m = 1$ in Theorem 3.4, we obtain the result presented in Theorem 3 of [3].
- (4) By setting $\alpha = s = 1$ in Theorem 3.4, we obtain the third result presented in Corollary 2 of [9].
- (5) By setting $\alpha = s = m = 1$ in Theorem 3.4, we obtain the result presented in Remark 13 of [14].
- (6) By setting $\alpha = m = 1$ in Theorem 3.4, we obtain the result presented in Theorem 10 of [14].
- (7) By setting $s \rightarrow 0$ with $\alpha = m = 1$ in Theorem 3.4, we obtain the second result presented in Corollary 2 of [8].

Corollary 3.6. Under the assumptions of the Theorem 3.4,

(1) If we put $p = q = 2$, then we get the result obtained by using Cauchy–Schwarz integral inequality as:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b - a)^2}{2(\alpha + 1)(s + 1)^{\frac{1}{2}}} \mathbf{B}^{\frac{1}{2}}(3, 2\alpha + 1) \left(\left[|f''(a)|^2 + m \left| f''\left(\frac{b}{m}\right) \right|^2 \right]^{\frac{1}{2}} \right. \\ & \quad \left. \left[m \left| f''\left(\frac{a}{m}\right) \right|^2 + |f''(b)|^2 \right]^{\frac{1}{2}} \right). \end{aligned}$$

(2) If we choose $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable (s, m) -convex function of 2nd kind:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(\zeta) d\zeta \right| \leq \frac{(b - a)^2}{4(s + 1)^{\frac{1}{q}}} \mathbf{B}^{\frac{1}{p}}(p + 1, p + 1) \times \\ & \left[\left(|f''(a)|^q + ms \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(ms \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

(3) If we choose $s \rightarrow 0$, then we get the following Fractional Hermite Hadamard type inequality for twice differentiable $P - m$ -convex function:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \frac{(b - a)^2}{2(\alpha + 1)} \mathbf{B}^{\frac{1}{p}}(p + 1, \alpha p + 1) \\ & \times \left[\left(|f''(a)|^q + m \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(m \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

(4) If we choose $s \rightarrow 0$ with $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable $P - m$ -convex function:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(\zeta) d\zeta \right| \leq \frac{(b - a)^2}{4} \mathbf{B}^{\frac{1}{p}}(p + 1, p + 1) \\ & \times \left[\left(|f''(a)|^q + m \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(m \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

(5) If we choose $s \rightarrow 0$ with $m = 1$, then we get the following Fractional Hermite Hadamard type inequality for twice differentiable P -convex function:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2}{\alpha + 1} \mathbf{B}^{\frac{1}{p}}(p+1, \alpha p + 1) (|f''(a)|^q + |f''(b)|^q)^{\frac{1}{q}}. \end{aligned}$$

Theorem 3.7. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a function that is twice differentiable on the interior of I , denoted by I° . Suppose $a, b \in I^\circ$ with $a < b$, and that $f'' \in L[a, b]$. If $|f''|^q$ is (s, m) -convex of the 2nd kind on I for some $s, m \in (0, 1]$ and $q \geq 1$, then the following inequality involving fractional integrals holds for any $\alpha > 0$:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)} \times \left[\left(\frac{2(\alpha+2)}{(s+2)(\alpha+s+2)} |f''(a)|^q + \right. \right. \\ & m \frac{2(\alpha+2)}{\alpha} (\mathbf{B}(2, s+1) - \mathbf{B}(\alpha+2, s+1)) \left| f'' \left(\frac{b}{m} \right) \right|^q \Big)^{\frac{1}{q}} \\ & + \left(m \frac{2(\alpha+2)}{\alpha} (\mathbf{B}(2, s+1) - \mathbf{B}(\alpha+2, s+1)) \left| f'' \left(\frac{a}{m} \right) \right|^q + \right. \\ & \left. \left. \frac{2(\alpha+2)}{(s+2)(\alpha+s+2)} |f''(b)|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The above outcome can be demonstrated through a method analogous to that used in the proof of Theorem 2.7, relying on the definition of (s, m) -convex functions of the 2nd kind. \square

Remark 3.8. The following well-established results can be obtained as special cases of our derived result by varying the values of s , m , and α :

- (1) By setting $s = 1$ in Theorem 3.7, we obtain the result presented in Theorem 1.11.
- (2) By setting $m = 1$ in Theorem 3.7, we obtain the result presented in Theorem 2.6 of [27].
- (3) By setting $s = m = 1$ in Theorem 3.7, we obtain the result presented in Theorem 4 of [3].
- (4) By setting $\alpha = s = 1$ in Theorem 3.7, we obtain the result presented in Corollary 3 of [9].

- (5) By setting $\alpha = s = m = 1$ in Theorem 3.7, we obtain the result presented in Remark 13 of [14].
- (6) By setting $\alpha = m = 1$ in Theorem 3.7, we obtain the result presented in Theorem 8 of [14].
- (7) By setting $s \rightarrow 0$ with $\alpha = m = 1$ in Theorem 3.7, we obtain the third result presented in Corollary 3 of [8].

Corollary 3.9. *Under the assumptions of the Theorem 3.7,*

- (1) *If we choose $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable (s, m) -convex function of 1st kind:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(\zeta) d\zeta \right| \leq \frac{(b-a)^2}{24} \left\{ \frac{6}{(s+2)(s+3)} \right\}^{\frac{1}{q}} \times \left[\left(|f''(a)|^q + \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right].$$

- (2) *If we choose $s \rightarrow 0$, then we get the following Fractional Hermite Hadamard type inequality for twice differentiable $P - m$ -convex function:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \frac{\alpha(b-a)^2}{4(\alpha + 1)(\alpha + 2)} \times \left[\left(|f''(a)|^q + m \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(m \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right].$$

- (3) *If we choose $s \rightarrow 0$ with $\alpha = 1$, then we get the following Hermite Hadamard type inequality for twice differentiable $P - m$ -convex function:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(\zeta) d\zeta \right| \leq \frac{(b-a)^2}{24} \times \left[\left(|f''(a)|^q + m \left| f''\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} + \left(m \left| f''\left(\frac{a}{m}\right) \right|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right].$$

- (4) *If we choose $s \rightarrow 0$ with $m = 1$, then we get the following Fractional Hermite Hadamard type inequality for twice differentiable P -convex function:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \right| \leq \frac{\alpha(b-a)^2}{2(\alpha + 1)(\alpha + 2)} (|f''(a)|^q + |f''(b)|^q)^{\frac{1}{q}}.$$

4. CONCLUSION

The fractional Hermite Hadamard inequality stands out as one of the most renowned within the realm of inequalities, boasting numerous generalizations across different classes of convex functions in existing literature. In this article, we present its extension for twice differentiable (s, m) -convex functions of first and 2^{nd} kinds. Section 2 and 3 unveils three distinct findings concerning the estimated right bound of the fractional Hermite Hadamard inequality in an absolute sense for twice differentiable (s, m) -convex functions of first and 2^{nd} kinds, respectively.

Here, we employ various methodologies, including Hölder's and Power mean integral inequalities. While some of these results are novel, others have been previously documented in articles such as [3, 8, 9, 10, 14, 22] and [27]. The final section is dedicated to providing remarks and offering future avenues of exploration for interested readers. Now, we are going to summarize the results of Section 2 and 3 in tabular form.

In the below two tables FHHTI, HHTI, CF and – stands for Fractional Hermite Hadamard type inequality, Hermite Hadamard type inequality, Convex Function and For any value, respectively.

S. No	m	s	α	Results	Found in
1	1	–	–	FHHTI for s -CF of 1^{st} kind	[3, 10]
2	–	1	–	FHHTI for m -CF	[9]
3	–	–	1	HHTI for (s, m) -CF of 1^{st} kind	This Article
4	1	1	–	FHHTI for CF	[3]
5	–	1	1	HHTI for m -CF	[9]
6	1	–	1	HHTI for s -CF of 1^{st} kind	[10, 22]
7	1	1	1	HHTI for CF	[8, 14, 22]
8	–	0	–	FHHTI for Quasi- m -CF	This Article
9	1	0	–	FHHTI for Quasi CF	This Article
10	–	0	1	HHTI for Quasi CF	This Article
11	1	0	1	HHTI for m -Convex Functions	[8]

Table 1. Result Summary of Section 2

S. No	m	s	α	Results	Found in
1	1	–	–	FHHTI for s -CF of 2^{nd} kind	[27]
2	–	1	–	FHHTI for m -CF	[9]
3	–	–	1	HHTI for (s, m) -CF of 2^{nd} kind	This Article
4	1	1	–	FHHTI for CF	[3]
5	–	1	1	HHTI for m -CF	[9]
6	1	–	1	HHTI for s -CF of 2^{nd} kind	[8, 14]
7	1	1	1	HHTI for CF	[14, 22]
8	–	0	–	FHHTI for $P - m$ -CF	This Article
9	1	0	–	FHHTI for P -CF	This Article
10	–	0	1	HHTI for P -CF	This Article
11	1	0	1	HHTI for m -Convex Functions	[8]

Table 2. Result Summary of Section 3

Now we are going to give some remarks and future ideas related to our stated results.

REMARKS AND FUTURE IDEAS

- (1) All the inequalities presented in this article can be reversed for concave functions, based on the fundamental equivalence: a function is concave if and only if its negative is convex.
- (2) The fractional Hermite-Hadamard inequality can be extended to a Fejér-type inequality by incorporating appropriate weighting functions.
- (3) Similar analyses can be carried out using various other generalized classes of convex functions.
- (4) The results established in this article may also be formulated in a discrete setting.
- (5) These results can be extended to the multi-dimensional case for broader applicability.
- (6) This study can be further developed within the framework of time scale calculus or quantum calculus.
- (7) An exploration of these results in the context of fuzzy theory offers another potential direction for future research.
- (8) One may also aim to derive sharper or more refined bounds for the inequalities established in this work.

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