

AN ELEMENTARY DERIVATION OF THE ORIENTED AREA FORMULA FOR PARAMETRIC SIMPLE CLOSED CURVES FROM THE SHOELACE THEOREM

IK-SEUNG LYOU^{a,*} AND HYEOKDEUK KWON^b AND SEUNGSOO KIM^c

ABSTRACT. This paper presents an elementary derivation of the oriented area formula

$$\text{Area}(D) = \frac{1}{2} \int_a^b (x(t)y'(t) - y(t)x'(t)) dt$$

for a parametrized simple closed curve. Instead of taking Green's theorem as a starting point, we construct inscribed polygons from a partition of the parameter interval, apply the Shoelace formula to each polygon, and pass to the limit through a Riemann-sum argument. This approach makes the geometric origin of the area integrand explicit and provides a bridge between polygonal area computation and line-integral area formulas. As an application, we compare two similar four-petal-type curves and verify that they have the same enclosed area while having slightly different perimeters.

1. INTRODUCTION

The Shoelace theorem (also called Gauss's area formula) gives polygonal area through coordinate determinants and is closely related to oriented area and cross products [9, 10]. In many textbook treatments, Green's theorem is presented first and the Shoelace formula is then derived as an application [4]. By contrast, expositions that reverse this direction—that is, deriving a Green-type area formula from Shoelace-based approximation—are relatively rare.

Motivated by this pedagogical gap, this paper derives an oriented area integral formula for parametrized simple closed curves from equal-interval approximation

Received by the editors April 15, 2026. Revised May 13, 2026. Accepted May 19, 2026.

2020 *Mathematics Subject Classification*. 97I50, 97G60, 97G60.

Key words and phrases. Shoelace formula, oriented area, simple closed curve, parametric curve, Green's theorem, Riemann sum.

*Corresponding author.

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(Riemann-sum intuition) and the Shoelace theorem [8], and then compares the result with the Green-theorem form [3, 4]. The goal is not to rederive the full Green theorem, but to establish its area-formula component by an elementary polygonal limit argument. From an educational perspective, this gives a concrete conceptual bridge from coordinate geometry (Shoelace formula) to vector-calculus area integrals [6, 7]. We begin with the triangle case obtained from the cross product. The magnitude of the cross product of two vectors $\vec{u} = \langle x_1, y_1 \rangle$, $\vec{v} = \langle x_2, y_2 \rangle$ is

$$|\vec{u} \times \vec{v}| = |x_1 y_2 - x_2 y_1|.$$

Therefore, for the area T of triangle OAB with vertices at the origin $O(0, 0)$ and points $A(x_1, y_1)$, $B(x_2, y_2)$, we have

$$T = \frac{1}{2} |\vec{OA} \times \vec{OB}| = \frac{1}{2} |x_1 y_2 - x_2 y_1|.$$

If the ordered triple O, A, B is counterclockwise, then $T = \frac{1}{2}(x_1 y_2 - x_2 y_1)$; if it is clockwise, then $T = -\frac{1}{2}(x_1 y_2 - x_2 y_1)$. This oriented-area interpretation extends naturally to convex n -gons.

Now assume that the vertices of a convex n -gon are listed in counterclockwise order. Let these points be

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n).$$

If we decompose the polygon into triangles with vertices O , (x_i, y_i) , and (x_{i+1}, y_{i+1}) , the total oriented area is

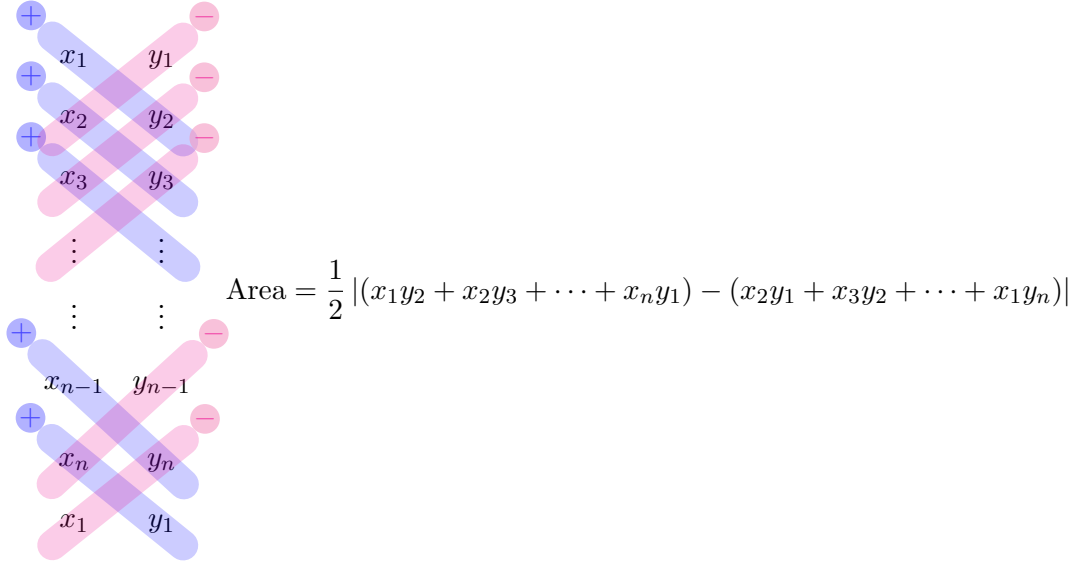
$$\text{Area} = \frac{1}{2} \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i).$$

Here, for cyclicity of indices, let $(x_{n+1}, y_{n+1}) = (x_1, y_1)$.

This identity is the sum of half cross products for the triangles formed with the origin. Negative terms correspond to clockwise orientation; thus the sum gives oriented area. The geometric area is the absolute value of this quantity.

$$\boxed{\text{Area} = \frac{1}{2} \left| \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i) \right|}$$

This formula is often computed efficiently using the following ‘‘Shoelace array’’:



Hence the name “Shoelace theorem.”

2. BACKGROUND: AREA FORMULA FROM GREEN’S THEOREM

Let C be a simple closed curve (piecewise C^1) traversed counterclockwise (positive direction), and let D be its interior region. If P, Q are C^1 on an open set containing D , then Green’s theorem gives [3, 4, 5]

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

The area S of D is

$$S = \iint_D 1 dx dy.$$

Therefore, choosing (P, Q) so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ yields

$$S = \oint_C P dx + Q dy.$$

Representative choices are:

- (1) $P = 0, Q = x \quad \Rightarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \quad \Rightarrow \quad \text{Area}(D) = \oint_C x dy,$
- (2) $P = -y, Q = 0 \quad \Rightarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \quad \Rightarrow \quad \text{Area}(D) = - \oint_C y dx,$
- (3) $P = -\frac{y}{2}, Q = \frac{x}{2} \quad \Rightarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \quad \Rightarrow \quad \text{Area}(D) = \frac{1}{2} \oint_C (x dy - y dx).$

Assume that C is given by a C^1 parametrization for $t \in [a, b]$

$$x = x(t), \quad y = y(t) \quad (t \in [a, b])$$

and that the orientation is counterclockwise. Substituting

$$dx = x'(t) dt, \quad dy = y'(t) dt$$

so the three formulas above become, respectively,

$$(1) \quad \text{Area}(D) = \oint_C x dy = \int_a^b x(t) y'(t) dt,$$

$$(2) \quad \text{Area}(D) = - \oint_C y dx = - \int_a^b y(t) x'(t) dt,$$

$$(3) \quad \text{Area}(D) = \frac{1}{2} \oint_C (x dy - y dx) = \frac{1}{2} \int_a^b (x(t) y'(t) - y(t) x'(t)) dt.$$

In particular, formulas (1)–(3) recover the usual definite integral formula for $y = f(x)$: Let $D = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$, where f is piecewise C^1 on $[a, b]$. C is the boundary of D , traversed counterclockwise, and can be divided into four segments: $C_1 : (a, 0) \rightarrow (b, 0)$, $C_2 : (b, 0) \rightarrow (b, f(b))$, $C_3 : (b, f(b)) \rightarrow (a, f(a))$ (along the graph), $C_4 : (a, f(a)) \rightarrow (a, 0)$. The integral for each segment is

$$\oint_{C_1} x dy = 0, \quad \oint_{C_2} x dy = bf(b)$$

and for the graph portion, $C_3 : (x, y) = (t, f(t))$, $t : b \rightarrow a$, so

$$\begin{aligned} \oint_{C_3} x dy &= \int_b^a t f'(t) dt = - \int_a^b t f'(t) dt \\ \oint_{C_4} x dy &= a(0 - f(a)) = -af(a). \end{aligned}$$

Therefore

$$\oint_C x dy = bf(b) - \int_a^b t f'(t) dt - af(a)$$

and applying integration by parts

$$\int_a^b t f'(t) dt = [tf(t)]_a^b - \int_a^b f(t) dt$$

we obtain

$$\oint_C x dy = bf(b) - \left([tf(t)]_a^b - \int_a^b f(t) dt \right) - af(a) = \int_a^b f(t) dt$$

and thus

$$\text{Area}(D) = \int_a^b f(x) dx.$$

Formulas (2) and (3) are verified analogously.

The next section presents the main contribution: a direct Shoelace-based derivation of formula (3), without invoking Green's theorem.

3. MAIN RESULT: SHOELACE-BASED DERIVATION

Let C be a simple closed curve (piecewise C^1), positively oriented and parametrized by

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b], \quad \gamma(a) = \gamma(b).$$

Let D denote the region enclosed by C .

For a partition

$$\mathcal{P}_n : a = t_0 < t_1 < \cdots < t_n = b,$$

define

$$\Delta t_k = t_{k+1} - t_k, \quad P_k = (x(t_k), y(t_k)), \quad \|\mathcal{P}_n\| = \max_k \Delta t_k.$$

Let Π_n be the inscribed polygon with consecutive vertices $P_0, P_1, \dots, P_{n-1}$ (and $P_n = P_0$).

Proposition 3.1 (Shoelace-to-Integral Area Formula). *Under the assumptions above,*

$$\text{Area}(D) = \frac{1}{2} \int_a^b (x(t)y'(t) - y(t)x'(t)) dt.$$

Proof. Applying the Shoelace formula to Π_n , its oriented area is

$$A_n = \frac{1}{2} \sum_{k=0}^{n-1} (x(t_k)y(t_{k+1}) - x(t_{k+1})y(t_k)).$$

Write

$$\Delta x_k = x(t_{k+1}) - x(t_k), \quad \Delta y_k = y(t_{k+1}) - y(t_k).$$

Then

$$A_n = \frac{1}{2} \sum_{k=0}^{n-1} (x(t_k)\Delta y_k - y(t_k)\Delta x_k) = \frac{1}{2} \sum_{k=0}^{n-1} \left(x(t_k) \frac{\Delta y_k}{\Delta t_k} - y(t_k) \frac{\Delta x_k}{\Delta t_k} \right) \Delta t_k.$$

By the mean value theorem, there exist $\xi_k, \eta_k \in [t_k, t_{k+1}]$ such that

$$\frac{\Delta y_k}{\Delta t_k} = y'(\xi_k), \quad \frac{\Delta x_k}{\Delta t_k} = x'(\eta_k).$$

Hence

$$A_n = \frac{1}{2} \sum_{k=0}^{n-1} \left(x(t_k) y'(\xi_k) - y(t_k) x'(\eta_k) \right) \Delta t_k.$$

Because x, y, x', y' are continuous on each smooth subinterval, this is a Riemann-sum form. [8] Therefore, as $\|\mathcal{P}_n\| \rightarrow 0$,

$$A_n \rightarrow \frac{1}{2} \int_a^b (x(t)y'(t) - y(t)x'(t)) dt.$$

For a piecewise C^1 Jordan curve, the oriented areas of inscribed polygons converge to the oriented area of the enclosed region. Thus $A_n \rightarrow \text{Area}(D)$, and the stated formula follows [4, 7]. $\square$

Remark 3.2. If the orientation is clockwise, the integral changes sign. The geometric area is the absolute value of the oriented area.

We now apply this result to a curve associated with ellipse normals.

4. APPLICATION: A CURVE ASSOCIATED WITH ELLIPSE NORMALS

Let

$$E_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0,$$

and let P be a point on $E_{a,b}$. Denote by l the normal line to the ellipse at P , and by Q the foot of the perpendicular from the center O to l .

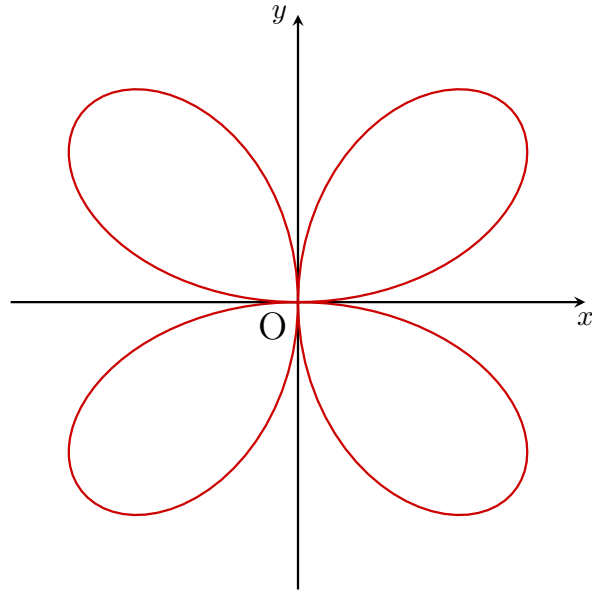
Proposition 4.1. *If $P = (a \cos \theta, b \sin \theta)$, then $Q = (x(\theta), y(\theta))$ is given by*

$$x(\theta) = \frac{a(a^2 - b^2) \sin^2 \theta \cos \theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}, \quad y(\theta) = \frac{-b(a^2 - b^2) \sin \theta \cos^2 \theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}.$$

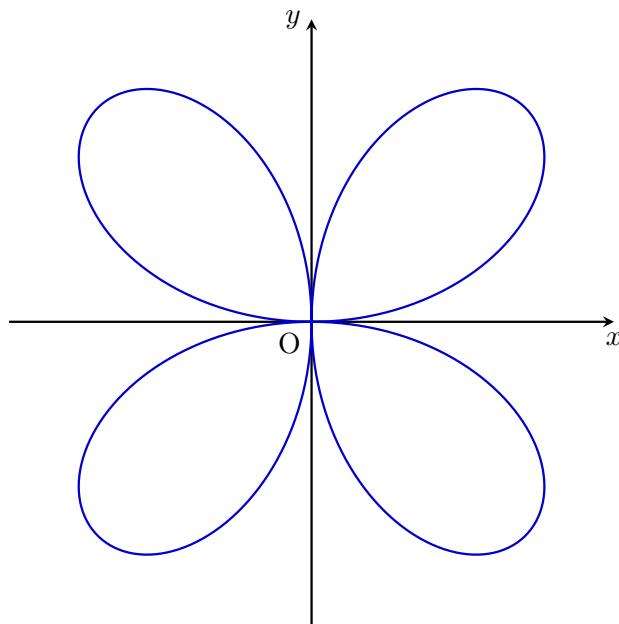
In this paper, we illustrate the computations with $(a, b) = (5, 4)$. Then

$$x(\theta) = \frac{45 \sin^2 \theta \cos \theta}{9 \sin^2 \theta + 16}, \quad y(\theta) = \frac{-36 \sin \theta \cos^2 \theta}{9 \sin^2 \theta + 16}.$$

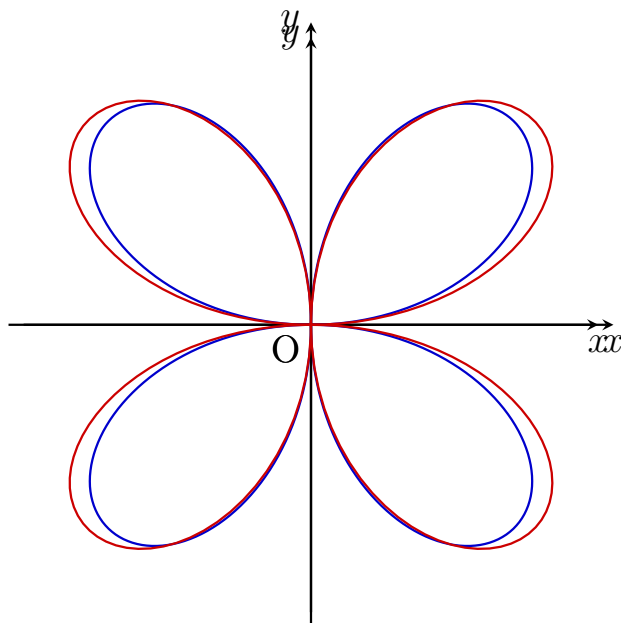
Graphing this for $0 \leq \theta \leq 2\pi$ gives the following:



For convenience, we denote this auxiliary curve by $\mathcal{C}_{5,4}$ (informally, the “butterfly-wing-type” curve). Its shape is close to that of the polar curve $r = \sin 2\theta$:



However, a closer inspection shows that the two curves are not identical. Overlaying them gives:



Thus, $\mathcal{C}_{5,4}$ is very close to, but distinct from, $r = \sin 2\theta$. We compare their enclosed areas and perimeters, starting with the polar curve area.

In polar coordinates, when the curve $r = f(\theta)$ traces a closed arc only once from $\theta = a$ to $\theta = b$, the area A is [5]

$$A = \frac{1}{2} \int_a^b (f(\theta))^2 d\theta.$$

For $r = \sin 2\theta$, zeros occur at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, and each interval traces one petal. In particular, on $[0, \frac{\pi}{2}]$, $r \geq 0$, so the curve starts at the origin, reaches $r = 1$, and returns to the origin.

Area of one petal A_{petal} :

$$\begin{aligned}
 A_{\text{petal}} &= \frac{1}{2} \int_0^{\pi/2} (\sin 2\theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2} d\theta \\
 &= \frac{1}{4} \int_0^{\pi/2} (1 - \cos 4\theta) d\theta \\
 &= \frac{1}{4} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}.
 \end{aligned}$$

Therefore, since the total area $A = 4A_{\text{petal}}$,

$$A = 4A_{\text{petal}} = 4 \cdot \frac{\pi}{8} = \frac{\pi}{2}$$

Next, we compute the enclosed area of $\mathcal{C}_{5,4}$. Using the area formula derived above, we first differentiate $x(\theta)$ by the quotient rule. Let the numerator and denominator of $x(\theta)$ be u_x and v_x :

$$\begin{aligned}
 u_x(\theta) &= 45 \sin^2 \theta \cos \theta \\
 v_x(\theta) &= 9 \sin^2 \theta + 16 \\
 u'_x(\theta) &= 90 \sin \theta \cos^2 \theta - 45 \sin^3 \theta \\
 v'_x(\theta) &= 18 \sin \theta \cos \theta \\
 x'(\theta) &= \frac{u'_x v_x - u_x v'_x}{v_x^2}
 \end{aligned}$$

Likewise, let the numerator and denominator of $y(\theta)$ be u_y and v_y :

$$\begin{aligned}
 u_y(\theta) &= -36 \sin \theta \cos^2 \theta \\
 v_y(\theta) &= 9 \sin^2 \theta + 16 \\
 u'_y(\theta) &= 72 \sin^2 \theta \cos \theta - 36 \cos^3 \theta \\
 v'_y(\theta) &= 18 \sin \theta \cos \theta \\
 y'(\theta) &= \frac{u'_y v_y - u_y v'_y}{v_y^2}
 \end{aligned}$$

Using these, compute the integrand:

$$\begin{aligned}
 x \cdot y' &= \frac{45 ((72 \sin^2 \theta \cos \theta - 36 \cos^3 \theta)(9 \sin^2 \theta + 16) + 648 \sin^2 \theta \cos^3 \theta) \sin^2 \theta \cos \theta}{(9 \sin^2 \theta + 16)^3} \\
 y \cdot x' &= -\frac{36 ((9 \sin^2 \theta + 16)(-45 \sin^3 \theta + 90 \sin \theta \cos^2 \theta) - 810 \sin^3 \theta \cos^2 \theta) \sin \theta \cos^2 \theta}{(9 \sin^2 \theta + 16)^3}
 \end{aligned}$$

Now

$$x \cdot y' - y \cdot x' = \frac{4860 \sin^6 \theta - 7695 \sin^4 \theta - 1215 \sin^2 \theta \cos \theta \cos(3\theta) + 2835 \sin^2 \theta}{81 \sin^4 \theta + 288 \sin^2 \theta + 256}.$$

Hence the desired area is

$$A = \frac{1}{2} \int_0^{2\pi} \left(\frac{4860 \sin^6 \theta - 7695 \sin^4 \theta - 1215 \sin^2 \theta \cos \theta \cos(3\theta) + 2835 \sin^2 \theta}{81 \sin^4 \theta + 288 \sin^2 \theta + 256} \right) d\theta.$$

To simplify the integrand, use substitutions. First, $\cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta$. Set $x = \sin^2 \theta$ ($0 \leq x \leq 1$), so $\cos^2 \theta = 1 - x$, and therefore

$$\begin{aligned} \sin^2 \theta \cos \theta \cos(3\theta) &= \sin^2 \theta \cos^2 \theta (4 \cos^2 \theta - 3) \\ &= x(1-x)(4(1-x) - 3) = x - 5x^2 + 4x^3. \end{aligned}$$

Therefore, the numerator of the integrand is

$$4860x^3 - 7695x^2 - 1215(x - 5x^2 + 4x^3) + 2835x = 1620x(1-x)$$

and the denominator is

$$81x^2 + 288x + 256 = (9x + 16)^2.$$

Therefore

$$f(\theta) = \frac{1620x(1-x)}{(9x+16)^2}, \quad x = \sin^2 \theta$$

so

$$A = \frac{1}{2} \int_0^{2\pi} f(\theta) d\theta = 2 \int_0^{\frac{\pi}{2}} f(\theta) d\theta = 3240 \int_0^{\frac{\pi}{2}} \frac{x(1-x)}{(9x+16)^2} d\theta.$$

For the remaining integral, let $t = \tan \theta$ ($0 \leq \theta \leq \pi/2$), then

$$x = \frac{t^2}{1+t^2} \quad 1-x = \frac{1}{1+t^2}$$

and $d\theta = \frac{dt}{1+t^2}$, $9x+16 = \frac{25t^2+16}{1+t^2}$. Thus, finding the enclosed area reduces to

$$A = 3240 \int_0^\infty \frac{t^2}{(1+t^2)(25t^2+16)^2} dt.$$

Expressing the integrand as partial fractions:

$$\begin{aligned} \frac{t^2}{(1+t^2)(25t^2+16)^2} &= \frac{C_1}{1+t^2} + \frac{C_2}{25t^2+16} + \frac{C_3}{(25t^2+16)^2}, \\ (C_1, C_2, C_3) &= \left(-\frac{1}{81}, \frac{25}{81}, -\frac{16}{9} \right) \end{aligned}$$

so

$$A = 3240 \left(-\frac{1}{81} \int_0^\infty \frac{dt}{1+t^2} + \frac{25}{81} \int_0^\infty \frac{dt}{25t^2+16} - \frac{16}{9} \int_0^\infty \frac{dt}{(25t^2+16)^2} \right)$$

and

$$\int_0^\infty \frac{dt}{1+t^2} = \frac{\pi}{2}, \quad \int_0^\infty \frac{dt}{25t^2+16} = \frac{\pi}{40}$$

are standard. For

$$I = \int_0^\infty \frac{dt}{(25t^2+16)^2}$$

rewrite the integrand as follows:

$$\begin{aligned} 25t^2 + 16 &= 25 \left(t^2 + \left(\frac{4}{5} \right)^2 \right) \\ \frac{1}{(25t^2+16)^2} &= \frac{1}{625} \cdot \frac{1}{(t^2+a^2)^2}, \quad a = \frac{4}{5}. \end{aligned}$$

Then

$$I = \frac{1}{625} \int_0^\infty \frac{dt}{(t^2+a^2)^2}, \quad a = \frac{4}{5}.$$

Let $J(a) = \int \frac{dt}{(t^2+a^2)^2}$ and substitute $t = a \tan \theta$, then $dt = a \sec^2 \theta d\theta$ and $t^2 + a^2 = a^2 \sec^2 \theta$. Therefore

$$\begin{aligned} J(a) &= \int \frac{a \sec^2 \theta d\theta}{a^4 \sec^4 \theta} = \frac{1}{a^3} \int \cos^2 \theta d\theta \\ &= \frac{1}{a^3} \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + C. \end{aligned}$$

Using $\theta = \arctan \left(\frac{t}{a} \right)$ and $\sin 2\theta = \frac{2ta}{t^2+a^2}$,

$$\int \frac{dt}{(t^2+a^2)^2} = \frac{1}{2a^3} \arctan \left(\frac{t}{a} \right) + \frac{t}{2a^2(t^2+a^2)} + C.$$

and therefore

$$\begin{aligned} \int_0^\infty \frac{dt}{(t^2+a^2)^2} &= \left[\frac{1}{2a^3} \arctan \left(\frac{t}{a} \right) + \frac{t}{2a^2(t^2+a^2)} \right]_0^\infty \\ &= \frac{1}{2a^3} \cdot \frac{\pi}{2} + 0 = \frac{\pi}{4a^3}. \end{aligned}$$

Hence

$$I = \frac{1}{625} \cdot \frac{\pi}{4a^3}, \quad a = \frac{4}{5}.$$

so

$$\int_0^\infty \frac{dt}{(25t^2+16)^2} = \frac{\pi}{1280}.$$

Therefore, the desired area is

$$A = 3240 \left(-\frac{1}{81} \cdot \frac{\pi}{2} + \frac{25}{81} \cdot \frac{\pi}{40} - \frac{16}{9} \cdot \frac{\pi}{1280} \right) = \frac{\pi}{2}.$$

Next, we compare the perimeters of $r = \sin 2\theta$ and $\mathcal{C}_{5,4}$. First, the arc length of the polar curve $r = \sin 2\theta$ is [5]

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Since

$$r(\theta) = \sin(2\theta), \quad \frac{dr}{d\theta} = 2 \cos(2\theta)$$

we have

$$L_{\text{petal}} = \int_0^{\pi/2} \sqrt{\sin^2(2\theta) + 4 \cos^2(2\theta)} d\theta = \int_0^{\pi/2} \sqrt{4 - 3 \sin^2(2\theta)} d\theta.$$

Using the substitution $\phi = 2\theta$ ($d\phi = 2 d\theta$) and symmetry,

$$\begin{aligned} L_{\text{petal}} &= \frac{1}{2} \int_0^\pi \sqrt{4 - 3 \sin^2 \phi} d\phi = \int_0^{\pi/2} \sqrt{4 - 3 \sin^2 \phi} d\phi \\ &= 2 \int_0^{\pi/2} \sqrt{1 - \frac{3}{4} \sin^2 \phi} d\phi = 2 E(k), \quad k^2 = \frac{3}{4}, \quad k = \frac{\sqrt{3}}{2} \end{aligned}$$

where

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi} d\phi$$

is Legendre's *complete elliptic integral (of the second kind)*. Therefore, the length of the entire curve (4 petals) is

$$L = 4 L_{\text{petal}} = 8 E\left(\frac{\sqrt{3}}{2}\right).$$

Now consider the total perimeter L of the parametric curve

$$x(\theta) = \frac{45 \sin^2 \theta \cos \theta}{9 \sin^2 \theta + 16}, \quad y(\theta) = \frac{-36 \sin \theta \cos^2 \theta}{9 \sin^2 \theta + 16}.$$

Let $D(\theta) = 9 \sin^2 \theta + 16$, $s = \sin \theta$, $c = \cos \theta$, then

$$\begin{aligned} x'(\theta) &= \frac{45(-9s^4 - 48s^2 + 32)s}{D^2}, \\ y'(\theta) &= \frac{36(9s^4 + 57s^2 - 16)c}{D^2}. \end{aligned}$$

Therefore, the velocity is

$$\sqrt{(x')^2 + (y')^2} = \frac{9\sqrt{9s^6 + 48s^4 - 48s^2 + 16}}{9s^2 + 16}.$$

The curve traces one petal (arc) for $\theta \in [0, \pi/2]$ and consists of 4 identical arcs in total, so

$$L = 4 L_{\text{petal}}, \quad L_{\text{petal}} = \int_0^{\pi/2} \frac{9\sqrt{9 \sin^6 \theta + 48 \sin^4 \theta - 48 \sin^2 \theta + 16}}{9 \sin^2 \theta + 16} d\theta.$$

Using the two substitutions $\phi = 2\theta$ and $t = \tan(\phi/2)$,

$$\cos \phi = \frac{1-t^2}{1+t^2}, \quad d\phi = \frac{2dt}{1+t^2}, \quad 0 \leq \theta \leq \frac{\pi}{2} \implies 0 \leq t \leq 1,$$

and by algebraic calculation,

$$L_{\text{petal}} = 9 \int_0^1 \frac{\sqrt{25t^6 + 16}}{(16 + 25t^2)(1+t^2)^{3/2}} dt.$$

Using the two substitutions

$$t^2 = \frac{16}{25} \frac{u}{1-u} \quad (0 \leq u \leq 1), \quad u = \sin^2 \psi \quad \left(0 \leq \psi \leq \frac{\pi}{2}\right)$$

the integral can be expressed as a linear combination of complete elliptic integrals of the second and third kinds.

$$k^2 = \frac{3}{4}, \quad n = -\frac{9}{16}.$$

Using the complete elliptic integrals of the second and third kinds:

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \psi} d\psi, \quad \Pi(n|k) = \int_0^{\pi/2} \frac{d\psi}{(1 - n \sin^2 \psi) \sqrt{1 - k^2 \sin^2 \psi}}$$

we can write

$$L_{\text{petal}} = A \cdot E(k) + B \cdot \Pi(n|k) \quad \left(k^2 = \frac{3}{4}, n = -\frac{9}{16}\right)$$

where A , B are rational constants determined by the corresponding partial-fraction decomposition. Finally, the total perimeter is

$$L = 4 L_{\text{petal}} = 4(A E(k) + B \Pi(n|k)), \quad k^2 = \frac{3}{4}, n = -\frac{9}{16}.$$

Closed forms in elementary functions are unavailable for both curves, so we evaluate them numerically:

$$\text{Polar curve length} \approx 9.6884482205, \quad \text{Curve } \mathcal{C}_{5,4} \approx 9.6907089907$$

Although the enclosed areas are equal, $\mathcal{C}_{5,4}$ has a slightly longer perimeter.

5. EDUCATIONAL IMPLICATIONS AND SCOPE

This paper is intended as a bridge example between school-level coordinate geometry and early university calculus. The present approach also reflects recent curricular emphases on mathematical reasoning, conceptual connections, and problem solving in school mathematics education [1, 2]. The key pedagogical point is that students who already know polygon area by determinants (Shoelace formula) can reach a nontrivial line-integral area identity through partition and limit reasoning,

before a full treatment of Green's theorem. In this sense, the argument supports conceptual continuity between:

- (1) polygonal area computation via the Shoelace formula (*high-school level*),
- (2) Riemann sums and definite integrals (*university level*),
- (3) oriented area for parametric closed curves via line integrals (*university level*).

The application section is presented as an illustrative case study rather than as a claim of discovering a new curve class.

6. CONCLUSION

We derived the oriented area formula

$$\text{Area}(D) = \frac{1}{2} \int_a^b (x(t)y'(t) - y(t)x'(t)) dt$$

for a parametrized simple closed curve directly from the Shoelace formula and a Riemann-sum limit. This establishes an elementary route to the area identity usually introduced as a consequence of Green's theorem. We then applied the formula to a curve associated with ellipse normals and compared it with $r = \sin 2\theta$, confirming equal enclosed areas in the studied case while observing a small perimeter difference.

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^aTEACHER, YANGHYEON HIGH SCHOOL: IK-SEUNG LYOU

Email address: `infgrp@hanmail.net`

^bTEACHER: HYEOKDEUK KWON, JEONJU GIRLS' HIGH SCHOOL

Email address: `wloghd@gmail.com`

^cTEACHER: SEUNGSOO KIM, JEONJU KIJUN GIRLS' HIGH SCHOOL

Email address: `prettyss@gmail.com`