

ON FUZZY PRE VOLTERRA SPACES

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ABSTRACT. This paper is focused on introducing and studying a new class of spaces called fuzzy pre Volterra spaces with the help of fuzzy pre dense and fuzzy pre G_δ -sets. Examples are given to illustrate the concepts and by using these concepts, to explore several interesting characterizations of fuzzy pre Volterra spaces in this paper. Also we examine the notions of fuzzy Volterra spaces and fuzzy pre Volterra spaces are independent to each other.

1. INTRODUCTION

The theory of fuzzy sets was discussed by L.A.Zadeh [20] in 1965. The notion of topological spaces in fuzzy setting was introduced and studied by C.L.Chang [7] in 1968. In General Topology, preopen sets, strongly semicontinuous functions and precontinuous functions were introduced by O.Njastad [9], A.S.Mashhour et. al. [8] and T.Noiri [10] respectively. The concept of fuzzy preopen sets was discussed by M.K.Singal and Niti Prakash [15] in 1991. Using fuzzy sets, the authors [19] introduced the concept of Volterra spaces. This paper aims to introduce and study fuzzy pre Volterra spaces using fuzzy pre dense and fuzzy pre G_δ -sets and hence to explore several interesting characterizations of fuzzy pre Volterra spaces.

The concept of fuzzy pre Volterra spaces may be applied to the study of open and closed fuzzy mappings via closure and interiors [14], generalized fuzzy topologies via fuzzy $g\mu$ -closed sets [13], fuzzy ideal topological spaces [2], fuzzy soft δ -separation and fuzzy soft δ -regularity axioms [1], stronger fuzzy pre-separation,

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regularity, symmetric and pre- R_i axioms [12], as well as δ -compactness and δ^* -compactness in fuzzy soft topological spaces using fuzzy soft δ -open sets [11].

Defining the fuzzy pre Volterra spaces enriches the theory of fuzzy topology by incorporating the fuzzy pre G_δ -set and fuzzy pre dense set. Such generalizations help bridge gaps between crisp and fuzzy sets and provide a foundation for further developments in generalized fuzzy topological spaces.

2. PRELIMINARIES

Now we recall basic results for the subsequent sections.

Definition 2.1. [7] The family ‘ T ’ of fuzzy sets or f-sets in short of a non-empty set X satisfies the following:

- (1) $0_X, 1_X \in T$,
- (2) If $A, B \in T$, then $A \cap B \in T$,
- (3) If $A_k \in T$ for each k , then $\cup A_k \in T$.

Then T is said to be a fuzzy topology or f-topology in short for X and the ordered pair (X, T) is a fuzzy topological space or fts in short and the members of T are said to be fuzzy open sets or fo sets in short. The complement of fo sets are fuzzy closed sets or fc sets in short.

Lemma 2.2. [3] For a family $\mathcal{A} = \{\alpha_k\}$ of f-sets of a fuzzy space X , Then, $\vee (cl(\alpha_k)) \leq cl(\vee \alpha_k)$. In case $\mathcal{A}$ is a finite set, $\vee cl(\alpha_k) = cl(\vee(\alpha_k))$. Also $\vee int(\alpha_k) \leq int(\vee \alpha_k)$.

Definition 2.3. [15] A f-set α in a fts is called a fuzzy pre open set or fpo set in short if $\alpha \leq int\ cl(\alpha)$.

Definition 2.4. [15] A f-set α in a fts is called a fuzzy pre closed set or fpc set in short if $cl\ int(\alpha) \leq \alpha$.

Lemma 2.5. [15] The fuzzy pre closure and the fuzzy pre interior of a f-set α in a fts are denoted and defined respectively as follows:

- (1) $pcl(\alpha) = \wedge \{\beta / \alpha \leq \beta \text{ and } \beta \text{ is a fpc set}\};$
- (2) $pint(\alpha) = \vee \{\beta / \beta \leq \alpha \text{ and } \beta \text{ is a fpo set}\}.$

Theorem 2.6. [15] Let α be a f-set of a fts. Then $1 - pint(\alpha) = pcl(1 - \alpha)$ and $1 - pcl(\alpha) = pint(1 - \alpha)$.

Definition 2.7. [4] A f-set α in a fts is called a fuzzy G_δ -set or fG_δ -set in short if $\alpha = \bigwedge_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fo sets.

Definition 2.8. [4] A f-set α in a fts is called a fuzzy F_σ -set or fF_σ -set in short if $\alpha = \bigvee_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fc sets.

Definition 2.9. [17] A f-set α in a fts is said to be a fuzzy dense set or fd set in short if there exists no fc set β such that $\alpha < \beta < 1$. That is, $cl(\alpha) = 1$.

Definition 2.10. [16] A f-set α in a fts is said to be a fuzzy pre dense set or fpd set in short if there exists no fpc set β such that $\alpha < \beta < 1$. That is, $pcl(\alpha) = 1$.

Definition 2.11. [16] A f-set α in a fts is said to be a fuzzy pre nowhere dense set or fpnd set in short if there exists no non-zero fpo set β such that $\beta < pcl(\alpha)$. That is, $pint\ pcl(\alpha) = 0$.

Definition 2.12. [16] A f-set α in a fts is said to be a fuzzy pre first category set or fpfc set in short if $\alpha = \bigvee_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fpnd sets. Any other f-set is a fuzzy pre second category set or fpsc set.

Definition 2.13. [16] Let α be a fpfc set in a fts. Then $1 - \alpha$ is called a fuzzy pre residual set or fpr set.

Definition 2.14. [18] A f-set α in a fts is called a fuzzy pre F_σ -set or fpF_σ -set in short if $\alpha = \bigvee_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fpc sets.

Definition 2.15. [18] A f-set α in a fts is called a fuzzy pre G_δ -set or fpG_δ -set in short if $\alpha = \bigwedge_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fpo sets.

Definition 2.16. [18] A f-set α in a fts is called a fuzzy pre σ -nowhere dense set or $fp\sigma$ -nd set in short if α is a nonzero fpF_σ -set such that $pint(\alpha) = 0$.

Definition 2.17. [18] A f-set α in a fts is called a fuzzy pre σ -first category set or $fp\sigma$ -fc set in short if $\alpha = \bigvee_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are $fp\sigma$ -nd sets. Any other f-set is said to be a fuzzy pre σ -second category set or $fp\sigma$ -sc set in short .

Definition 2.18. [18] Let α be a $fp\sigma$ -fc set in a fts. Then $1 - \alpha$ is called a fuzzy pre σ -residual set or $fp\sigma$ -r set in short.

Definition 2.19. [19] A fts is said to be a fuzzy Volterra space or fVs in short if $cl\left(\bigwedge_{k=1}^N(\alpha_k)\right) = 1$, where (α_k) 's are fd and fG_δ -sets.

Theorem 2.20. [6] *For a f -set α in a fts , the following are true:*

- (1) $pcl(\alpha) \geq \alpha \vee cl\ int(\alpha)$.
- (2) $pint(\alpha) \leq \alpha \wedge int\ cl(\alpha)$.

Theorem 2.21. [6] *For a f -set α in a fts , the following properties hold:*

- (1) $int\ pcl(\alpha) \leq int\ cl(\alpha)$.
- (2) $int\ pcl(\alpha) \geq int\ cl\ (int(\alpha))$.

Theorem 2.22. [6] *Let α be a f -set in a fts . Then $int(\alpha) \leq pint(\alpha) \leq \alpha \leq pcl(\alpha) \leq cl(\alpha)$.*

Theorem 2.23. [5] *Let α and β be f -sets of a fts . Then*

- (1) $pcl(\alpha \vee \beta) \geq pcl(\alpha) \vee pcl(\beta)$;
- (2) $pcl(\alpha \wedge \beta) \leq pcl(\alpha) \wedge pcl(\beta)$;
- (3) $pint(\alpha \vee \beta) \geq pint(\alpha) \vee pint(\beta)$;
- (4) $pint(\alpha \wedge \beta) \leq pint(\alpha) \wedge pint(\beta)$.

Theorem 2.24. [5] *A f -set α in a fts is called a fpo set if $\alpha = pint(\alpha)$ and fpc set if $\alpha = pcl(\alpha)$.*

Theorem 2.25. [16] *If α is a $fpnd$ set in a fts , then $1 - \alpha$ is a fpd set.*

Theorem 2.26. [18] *A f -set α in a fts is a $fp\sigma$ -nd set if and only if $1 - \alpha$ is a fpd and fpG_δ -set.*

3. FUZZY PRE DENSE AND FUZZY PRE G_δ -SETS

Proposition 3.1. *A f -set α is a fpo set in a fts if and only if $1 - \alpha$ is a fpc set.*

Proof. Let α denotes a fpo set. Then $\alpha \leq int\ cl(\alpha)$. Now $cl\ int(1 - \alpha) = 1 - int\ cl(\alpha) \leq 1 - \alpha$. So, $1 - \alpha$ is a fpc set.

Conversely, let α denotes a fpc set. Then $cl\ int(\alpha) \leq \alpha$. Now $int\ cl(1 - \alpha) = 1 - cl\ int(\alpha) \geq 1 - \alpha$. So, $1 - \alpha$ is a fpo set. $\square$

Proposition 3.2. *If α is a fpd set in a fts , then α is a fd set.*

Proof. Let α be a fpd set. Then $pcl(\alpha) = 1$. Since $pcl(\alpha) \leq cl(\alpha)$, $1 \leq cl(\alpha)$. That is, $cl(\alpha) = 1$. So, α is a fd set. $\square$

Remark 3.3. From the above Proposition 3.2, the converse need not be true. That is, a fd set need not be a fpd set. For consider the following example:

The following f-sets are defined on $X = \{a, b, c\}$ as follows:

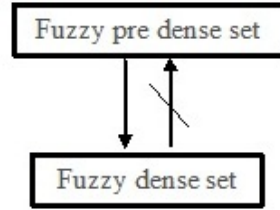
$$\beta_1 = \{(a, 0.5), (b, 0.4), (c, 0.6)\};$$

$$\beta_2 = \{(a, 0.6), (b, 0.5), (c, 0.7)\};$$

$$\beta_3 = \{(a, 0.3), (b, 0.1), (c, 0.7)\}.$$

Clearly $T = \{0, \beta_1, \beta_2, 1\}$ is a f-topology on X . By computations, one can see that $cl(\beta_3) = 1$ but $pcl(\beta_3) = \beta_3 \neq 1$. Hence β_3 is a fd set but not a fpd set.

Diagrammatically,



Proposition 3.4. If α is a fG_δ -set in a fts, then α is a fpG_δ -set.

Proof. Let α denotes a fG_δ -set. Then $\alpha = \bigwedge_{k=1}^{\infty} (\alpha_k)$, where (α_k) 's are fo sets. Since fo sets (α_k) 's are fpo sets, $\alpha = \bigwedge_{k=1}^{\infty} (\alpha_k)$ where (α_k) 's are fpo sets and hence α is a fpG_δ -set. $\square$

Remark 3.5. The converse of the above Proposition need not be true. For consider the following example:

Example 3.6. The following f-sets are defined on $X = \{a, b, c\}$ as follows:

$$\beta_1 = \{(a, 0.8), (b, 0.6), (c, 0.7)\};$$

$$\beta_2 = \{(a, 0.6), (b, 0.9), (c, 0.8)\};$$

$$\beta_3 = \{(a, 0.7), (b, 0.5), (c, 0.9)\}.$$

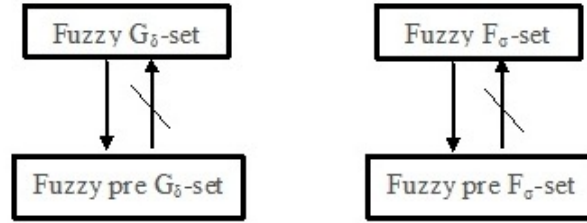
Then, $T = \{0, \beta_1, \beta_2, \beta_3, \beta_1 \vee \beta_2, \beta_1 \vee \beta_3, \beta_2 \vee \beta_3, \beta_1 \wedge \beta_2, \beta_1 \wedge \beta_3, \beta_2 \wedge \beta_3, \beta_1 \wedge (\beta_2 \vee \beta_3), \beta_1 \vee (\beta_2 \wedge \beta_3), \beta_2 \wedge (\beta_1 \vee \beta_3), \beta_2 \vee (\beta_1 \wedge \beta_3), \beta_3 \wedge (\beta_1 \vee \beta_2), \beta_3 \vee (\beta_1 \wedge \beta_2), \beta_1 \vee \beta_2 \vee \beta_3, 1\}$ is a f-topology on X . Now the fuzzy set $\alpha = \{[\beta_1 \vee (\beta_2 \wedge \beta_3)] \wedge [\beta_3 \vee (\beta_1 \wedge \beta_3)] \wedge [\beta_3 \vee (\beta_1 \wedge \beta_2)]\}$ is a fpG_δ -set in (X, T) . But α is not a fG_δ -set in (X, T) since every fo set is a fpo set but the converse need not be true.

Proposition 3.7. If α is a fF_σ -set in a fts, then α is a fpF_σ -set.

Proof. Let α denotes a fF_σ -set. Then $\alpha = \bigvee_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fc sets. Since fc sets (α_k) 's are fpc sets, $\alpha = \bigvee_{k=1}^{\infty}(\alpha_k)$ where (α_k) 's are fpc sets and hence α is a fpF_σ -set. $\square$

Remark 3.8. From the above two Propositions 3.4 and 3.7, one can see that the converse need not be true. That is, “If α is a fpG_δ -set in a fts, then α need not be a fG_δ -set [since fo sets are fpo sets but the converse need not be true]” and “if α is a fpF_σ -set in a fts, then α need not be a fF_σ -set [since fc sets are fpc sets but the converse need not be true]”.

Diagrammatically,



Proposition 3.9. *A f-set α is a fpG_δ -set in a fts if and only if $1 - \alpha$ is a fpF_σ -set.*

Proof. Let α denotes a fpG_δ -set. Then $\alpha = \bigwedge_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fpo sets. Since (α_k) 's are fpo sets, $(1 - \alpha_k)$'s are fpc sets. Now $1 - \alpha = 1 - \bigwedge_{k=1}^{\infty}(\alpha_k) = \bigvee_{k=1}^{\infty}(1 - \alpha_k)$. So, $1 - \alpha$ is a fpF_σ -set.

Conversely, let α denotes a fpF_σ -set. Then $\alpha = \bigvee_{k=1}^{\infty}(\alpha_k)$, where (α_k) 's are fpc sets. Since (α_k) 's are fpc sets, $(1 - \alpha_k)$'s are fpo sets. Now $1 - \alpha = 1 - \bigvee_{k=1}^{\infty}(\alpha_k) = \bigwedge_{k=1}^{\infty}(1 - \alpha_k)$. So, $1 - \alpha$ is a fpG_δ -set. $\square$

Proposition 3.10. *If α is a fpd and fpo set in a fts, then $1 - \alpha$ is a fpnd set.*

Proof. Let α denotes a fpd and fpo set. Then $pcl(\alpha) = 1$ and $pint(\alpha) = \alpha$. Now $pint\ pcl(1 - \alpha) = 1 - pcl(pint(\alpha)) = 1 - pcl(\alpha) = 1 - 1 = 0$. So, $1 - \alpha$ is a fpnd set. $\square$

Proposition 3.11. *If $\alpha \leq \beta$ and β is a fpnd set in a fts, then α is also a fpnd set.*

Proof. Let β denotes a fpnd set. Then, $pint\ pcl(\beta) = 0$. Since $\alpha \leq \beta$, $pint\ pcl(\alpha) \leq pint\ pcl(\beta)$. This implies that $pint\ pcl(\alpha) \leq 0$. That is, $pint\ pcl(\alpha) = 0$. So, α is also a fpnd set. $\square$

Proposition 3.12. *If $\alpha = \beta \vee \delta$, where β is a fpd and fpo set and δ is a fpnd set in a fts, then the following are hold:*

- (i) $1 - \alpha$ is a fpnd set.
- (ii) $\text{pint}(1 - \alpha) = 0$.
- (iii) $1 - \alpha$ is not a fpd set.

Proof. (i). Let β denotes a fpd and fpo set. Then $\text{pcl}(\beta) = 1$ and $\text{pint}(\beta) = \beta$. Now $\text{pint } \text{pcl}(1 - \beta) = 1 - \text{pcl}(\text{pint}(\beta)) = 1 - \text{pcl}(\beta) = 1 - 1 = 0$ and hence $1 - \beta$ is a fpnd set. Since $\alpha = \beta \vee \delta$, $\alpha \geq \beta$ or $\alpha \geq \delta$. Consider $\alpha \geq \beta$. Then $1 - \alpha \leq 1 - \beta$. By the Proposition 3.11, $1 - \alpha$ is a fpnd set.

(ii). Let $1 - \alpha$ denotes a fpnd set. Then, $\text{pint } \text{pcl}(1 - \alpha) = 0$. Since $\text{pint}(1 - \alpha) \leq \text{pint } \text{pcl}(1 - \alpha)$, $\text{pint}(1 - \alpha) \leq 0$. That is, $\text{pint}(1 - \alpha) = 0$.

(iii). Let δ denotes a fpnd set. Then, $\text{pint } \text{pcl}(\delta) = 0$. Since $\text{pint}(\delta) \leq \text{pint } \text{pcl}(\delta)$, $\text{pint}(\delta) \leq 0$. That is, $\text{pint}(\delta) = 0$. Since $\alpha = \beta \vee \delta$, $\alpha \geq \beta$ or $\alpha \geq \delta$. Consider $\alpha \geq \delta$. Now $\text{pint}(\alpha) \geq \text{pint}(\delta)$. Then $\text{pint}(\alpha) \geq 0$. This implies that $1 - \text{pint}(\alpha) \leq 1$. This will imply that $\text{pcl}(1 - \alpha) \leq 1 \neq 1$. So, $1 - \alpha$ is not a fpd set. $\square$

Proposition 3.13. *A f-set α is a fpnd set in a fts if and only if $1 - \text{pcl}(\alpha)$ is a fpo and fpd set.*

Proof. Let α denotes a fpnd set. Then, $\text{pint } \text{pcl}(\alpha) = 0$. Now $\text{pcl}(\alpha)$ is a fpc set. Then $1 - \text{pcl}(\alpha)$ is a fpo set. Also $\text{pcl}[1 - \text{pcl}(\alpha)] = 1 - \text{pint } \text{pcl}(\alpha) = 1 - 0 = 1$. Then $1 - \text{pcl}(\alpha)$ is a fpd set. Thus $1 - \text{pcl}(\alpha)$ is a fpo and fpd set.

Conversely, let $1 - \text{pcl}(\alpha)$ denotes a fpo and fpd set. Then $\text{pcl}[1 - \text{pcl}(\alpha)] = 1$. Now $1 - \text{pint } \text{pcl}(\alpha) = 1$, implies that $\text{pint } \text{pcl}(\alpha) = 0$. So, α is a fpnd set. $\square$

Proposition 3.14. *If $\alpha = \bigvee_{k=1}^{\infty} (\alpha_k)$, where $(1 - \alpha_k)$'s are fpo sets, is a fpnd set in a fts, then (α_k) 's are fpnd sets.*

Proof. Let $\alpha = \bigvee_{k=1}^{\infty} (\alpha_k)$, where $(1 - \alpha_k)$'s are fpo sets. Then $\text{pcl}(\alpha) = \text{pcl}(\bigvee_{k=1}^{\infty} (\alpha_k)) \geq \bigvee_{k=1}^{\infty} (\text{pcl}(\alpha_k))$. Since $(1 - \alpha_k)$'s are fpo sets, (α_k) 's are fpc sets. Then $\text{pcl}(\alpha_k) = \alpha_k \rightarrow (1)$. Hence $\text{pcl}(\alpha) \geq \bigvee_{k=1}^{\infty} (\alpha_k)$. Then $\text{pint } \text{pcl}(\alpha) \geq \text{pint}(\bigvee_{k=1}^{\infty} (\alpha_k)) \geq \bigvee_{k=1}^{\infty} (\text{pint}(\alpha_k))$. Since α is a fpnd set, $\text{pint } \text{pcl}(\alpha) = 0$. Thus $0 \geq \bigvee_{k=1}^{\infty} (\text{pint}(\alpha_k))$. That is, $\bigvee_{k=1}^{\infty} (\text{pint}(\alpha_k)) = 0$. Hence $\text{pint}(\alpha_k) = 0$ for each k . Now, $\text{pint } \text{pcl}(\alpha_k) = \text{pint}(\alpha_k) = 0$ [from (1)]. So, (α_k) 's are fpnd sets. $\square$

Proposition 3.15. *If α is a fpd and fpG_δ -set in a fts, then $1 - \alpha$ is a fpfc set.*

Proof. Let α denotes a fpd and fpG_δ -set. Then $pcl(\alpha) = 1$ and $\alpha = \bigwedge_{k=1}^\infty (\alpha_k)$, where (α_k) 's are fpo sets. This implies that $pcl(\bigwedge_{k=1}^\infty (\alpha_k)) = 1$. But $pcl(\bigwedge_{k=1}^\infty (\alpha_k)) \leq \bigwedge_{k=1}^\infty (pcl(\alpha_k))$. Hence $1 \leq \bigwedge_{k=1}^\infty (pcl(\alpha_k))$. That is, $\bigwedge_{k=1}^\infty (pcl(\alpha_k)) = 1$ and hence $pcl(\alpha_k) = 1$. Since (α_k) 's are fpo sets, $pint(\alpha_k) = \alpha_k$ and therefore $pcl(pint(\alpha_k)) = 1$. Now $1 - (pcl(pint(\alpha_k))) = 0$, implies that $pint(pcl(1 - \alpha_k)) = 0$. Thus $(1 - \alpha_k)$'s are fpnd sets. Then $1 - \alpha = 1 - \bigwedge_{k=1}^\infty (\alpha_k) = \bigvee_{k=1}^\infty (1 - \alpha_k)$. So, $1 - \alpha$ is a fpfc set. $\square$

Proposition 3.16. *If α is a fpd and fpG_δ -set in a fts, then α is a fpr set.*

Proof. Let α denotes a fpd and fpG_δ -set. Then by the Proposition 3.15, $1 - \alpha$ is a fpfc set. Therefore, α is a fpr set. $\square$

Proposition 3.17. *If a fpnd set α in a fts is a fpF_σ -set, then α is a fpfc set.*

Proof. Let α denotes a fpnd set such that $\alpha = \bigvee_{k=1}^\infty (\alpha_k)$, where $(1 - \alpha_k)$'s are fpo sets. Since α is a fpnd set and by the Proposition 3.14, (α_k) 's are fpnd sets. Then $\alpha = \bigvee_{k=1}^\infty (\alpha_k)$, where (α_k) 's are fpnd sets. So, α is a fpfc set. $\square$

Proposition 3.18. *If $pint(\alpha) = 0$ for a fpF_σ -set α in a fts, then α is a fpfc set.*

Proof. Let α denotes a fpF_σ -set such that $pint(\alpha) = 0$. Then $\alpha = \bigvee_{k=1}^\infty (\alpha_k)$, where (α_k) 's are fpc sets. This implies that $0 = pint(\alpha) = pint(\bigvee_{k=1}^\infty (\alpha_k))$. But $\bigvee_{k=1}^\infty (pint(\alpha_k)) \leq pint(\bigvee_{k=1}^\infty (\alpha_k)) = 0$ $\bigvee_{k=1}^\infty (pint(\alpha_k)) \leq 0$. That is, $\bigvee_{k=1}^\infty (pint(\alpha_k)) = 0$. Then $pint(\alpha_k) = 0$ for each k . Since (α_k) 's are fpc sets, $pcl(\alpha_k) = \alpha_k$. Now $pint(pcl(\alpha_k)) = pint(\alpha_k) = 0$. Thus (α_k) 's are fpnd sets. Therefore $\alpha = \bigvee_{k=1}^\infty (\alpha_k)$, where (α_k) 's are fpnd sets. So, α is a fpfc set. $\square$

Proposition 3.19. *If a fpG_δ -set α is a fpd set in a fts, then α is a fpr set.*

Proof. Let α denotes a fpG_δ -set such that $pcl(\alpha) = 1$. Then $1 - \alpha$ is a fpF_σ -set such that $1 - pcl(\alpha) = 0$. That is, $1 - \alpha$ is a fpF_σ -set such that $pint(1 - \alpha) = 0$. Then by the Proposition 3.18, $1 - \alpha$ is a fpfc set. So, α is a fpr set. $\square$

Proposition 3.20. *If α is a fpfc set in a fts, then $1 - \alpha = \bigwedge_{k=1}^\infty (\beta_k)$, where $pcl(\beta_k) = 1$.*

Proof. Let α denotes a fpfc set. Then $\alpha = \bigwedge_{k=1}^\infty (\alpha_k)$, where (α_k) 's are fpnd sets. Now $1 - \alpha = 1 - \bigwedge_{k=1}^\infty (\alpha_k) = \bigvee_{k=1}^\infty (1 - \alpha_k)$. Since (α_k) 's are fpnd sets and by the Theorem 2.25, $(1 - \alpha_k)$'s are fpd sets. That is, $pcl(1 - \alpha_k) = 1$. Let $\beta_k = 1 - \alpha_k$. Thus, $1 - \alpha = \bigwedge_{k=1}^\infty (\beta_k)$, where $pcl(\beta_k) = 1$. $\square$

4. FUZZY PRE VOLTERRA SPACES

Definition 4.1. A fts is called a fuzzy pre Volterra space or fpVs in short if $cl(\bigwedge_{k=1}^N(\alpha_k)) = 1$, where (α_k) 's are fpd and fpG_δ -sets.

Example 4.2. The following f-sets are defined on $X = \{a, b, c\}$ as follows:

$$\alpha_1 = \{(a, 0.8), (b, 0.6), (c, 0.7)\};$$

$$\alpha_2 = \{(a, 0.6), (b, 0.9), (c, 0.8)\};$$

$$\alpha_3 = \{(a, 0.7), (b, 0.5), (c, 0.9)\}.$$

Clearly $T = \{0, \alpha_1, \alpha_2, \alpha_3, \alpha_1 \vee \alpha_2, \alpha_1 \vee \alpha_3, \alpha_2 \vee \alpha_3, \alpha_1 \wedge \alpha_2, \alpha_1 \wedge \alpha_3, \alpha_2 \wedge \alpha_3, \alpha_1 \vee (\alpha_2 \wedge \alpha_3), \alpha_1 \wedge (\alpha_2 \vee \alpha_3), \alpha_2 \vee (\alpha_1 \wedge \alpha_3), \alpha_2 \wedge (\alpha_1 \vee \alpha_3), \alpha_3 \vee (\alpha_1 \wedge \alpha_2), \alpha_3 \wedge (\alpha_1 \vee \alpha_2), \alpha_1 \vee \alpha_2 \vee \alpha_3, 1\}$ is a f-topology on X .

Now consider the following f-sets:

$$\beta_1 = \{\alpha_2 \wedge (\alpha_1 \vee \alpha_2) \wedge (\alpha_2 \wedge \alpha_3) \wedge [\alpha_2 \vee (\alpha_1 \wedge \alpha_3)] \wedge [\alpha_3 \wedge (\alpha_1 \vee \alpha_2)] \wedge [\alpha_2 \wedge (\alpha_1 \vee \alpha_3)] \wedge [\alpha_1 \vee (\alpha_2 \wedge \alpha_3)]\}$$

$$\beta_2 = \{\alpha_1 \wedge (\alpha_1 \wedge \alpha_2) \wedge (\alpha_1 \wedge \alpha_3) \wedge [\alpha_1 \wedge (\alpha_2 \vee \alpha_3)]\} \text{ and}$$

$$\beta_3 = \{\alpha_3 \wedge (\alpha_2 \vee \alpha_3) \wedge (\alpha_1 \vee \alpha_3) \wedge [\alpha_1 \vee (\alpha_2 \wedge \alpha_3)] \wedge [\alpha_3 \vee (\alpha_1 \wedge \alpha_2)]\}.$$

Then β_1, β_2 and β_3 are fG_δ -sets and hence are fpG_δ -sets. On computations, one can see that $pcl(\beta_1) = 1$, $pcl(\beta_2) = 1$ and $pcl(\beta_3) = 1$. Hence β_1, β_2 and β_3 are fpd sets. This implies that $pcl(\beta_1 \wedge \beta_2 \wedge \beta_3) = 1$, where $\beta_1, \beta_2, \beta_3$ are fpd and fpG_δ -sets. So, (X, T) is a fpVs.

Example 4.3. The following f-sets are defined on $X = \{a, b, c\}$ as follows:

$$\alpha_1 = \{(a, 0.4), (b, 0.5), (c, 0.6)\};$$

$$\alpha_2 = \{(a, 0.6), (b, 0.4), (c, 0.5)\};$$

$$\alpha_3 = \{(a, 0.7), (b, 0.6), (c, 0.4)\}.$$

Then, $T = \{0, \alpha_1, \alpha_2, \alpha_3, \alpha_1 \vee \alpha_2, \alpha_1 \vee \alpha_3, \alpha_2 \vee \alpha_3, \alpha_1 \wedge \alpha_2, \alpha_1 \wedge \alpha_3, \alpha_2 \wedge \alpha_3, \alpha_1 \wedge (\alpha_2 \vee \alpha_3), \alpha_2 \wedge (\alpha_1 \wedge \alpha_3), \alpha_3 \wedge (\alpha_1 \vee \alpha_2), 1\}$ is a f-topology on X . Now there is no fpd and fpG_δ -set such that $pcl(\bigwedge_{k=1}^N(\alpha_k)) = 1$ and therefore (X, T) is not a fpVs.

5. CHARACTERIZATIONS OF FUZZY PRE VOLTERRA SPACES

Proposition 5.1. If $pint(\bigvee_{k=1}^N(\alpha_k)) = 0$, where (α_k) 's are fpnd and fpF_σ -sets in (X, T) , then (X, T) is a fpVs.

Proof. Suppose that $pint(\bigvee_{k=1}^N(\alpha_k)) = 0$. Then $1 - pint(\bigvee_{k=1}^N(\alpha_k)) = 1$ and $pcl(\bigwedge_{k=1}^N(1 - \alpha_k)) = 1$. Since (α_k) 's are fpnd sets and by the Theorem 2.25,

$(1 - \alpha_k)$'s are fpd sets. Also since (α_k) 's are $\text{fp}F_\sigma$ -sets and by the Proposition 3.9, $(1 - \alpha_k)$'s are $\text{fp}G_\delta$ -sets. This implies that $\text{pcl}(\bigwedge_{k=1}^N(1 - \alpha_k)) = 1$, where $(1 - \alpha_k)$'s are fpd and $\text{fp}G_\delta$ -sets. Therefore, (X, T) is a fpVs. $\square$

Proposition 5.2. *A (X, T) is a fpVs if and only if $\text{pint}(\bigvee_{k=1}^N(1 - \alpha_k)) = 0$, where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets.*

Proof. Let (X, T) denotes a fpVs and (α_k) 's denote fpd and $\text{fp}G_\delta$ -sets. Then $\text{pcl}(\bigwedge_{k=1}^N(\alpha_k)) = 1$. Now $\text{pint}(\bigvee_{k=1}^N(1 - \alpha_k)) = 1 - \text{pcl}(\bigwedge_{k=1}^N(\alpha_k)) = 1 - 1 = 0$. Hence $\text{pint}(\bigvee_{k=1}^N(1 - \alpha_k)) = 0$, where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets.

Conversely, let $\text{pint}(\bigvee_{k=1}^N(1 - \alpha_k)) = 0$ where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets. This implies that $1 - \text{pcl}(\bigwedge_{k=1}^N(\alpha_k)) = 0$ and hence $\text{pcl}(\bigwedge_{k=1}^N(\alpha_k)) = 1$, where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets. So, (X, T) is a fpVs. $\square$

Proposition 5.3. *If the f-sets (β_k) 's, $(i = 1 \text{ to } N)$ are fpfc sets formed from the fpd and $\text{fp}G_\delta$ -sets in a fpVs, then $\text{pint}(\bigvee_{k=1}^N(\beta_k)) = 0$.*

Proof. Let (α_k) 's denote the fpd and $\text{fp}G_\delta$ -sets. Then $\text{pcl}(\bigwedge_{k=1}^N(\alpha_k)) = 1$. Now $1 - \text{pcl}(\bigwedge_{k=1}^N(\alpha_k)) = 0$. This implies that $\text{pint}(\bigwedge_{k=1}^N(1 - \alpha_k)) = 0$. Since the f-sets (α_k) 's are fpd and $\text{fp}G_\delta$ -sets and by the Proposition 3.15, $(1 - \alpha_k)$'s are fpfc sets. Let $\beta_k = 1 - \alpha_k$. Hence $\text{pint}(\bigvee_{k=1}^N(\beta_k)) = 0$, where (β_k) 's are fpfc sets. $\square$

Proposition 5.4. *If $\alpha = \bigwedge_{k=1}^N(\alpha_k)$, where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets in a fpVs (X, T) , then α is not a fpc set.*

Proof. Let $\alpha = \bigwedge_{k=1}^N(\alpha_k)$, where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets. Since (X, T) is a fpVs, $\text{pcl}(\bigwedge_{k=1}^N(\alpha_k)) = 1$. That is, $\text{pcl}(\alpha) = 1 \neq \alpha$. Therefore, α is not a fpc set. $\square$

Proposition 5.5. *Let $\beta = \bigvee_{k=1}^N(\beta_k)$, where (β_k) 's are fpnd and $\text{fp}F_\sigma$ -sets in a fpVs. Then β is not a fpo set.*

Proof. Now $\beta = \bigvee_{k=1}^N(\beta_k)$, where (β_k) 's are fpnd and $\text{fp}F_\sigma$ -sets. This implies that $1 - \beta = 1 - \bigvee_{k=1}^N(\beta_k) = \bigwedge_{k=1}^N(1 - \beta_k)$. Since (β_k) 's are fpnd sets and by the Theorem 2.25, $(1 - \beta_k)$'s are fpd sets. Also since (β_k) 's are $\text{fp}F_\sigma$ -sets and by the Proposition 3.9, $(1 - \beta_k)$'s are $\text{fp}G_\delta$ -sets. Hence $1 - \beta = \bigwedge_{k=1}^N(1 - \beta_k)$, where $(1 - \beta_k)$'s are fpd and $\text{fp}G_\delta$ -sets. Then by the Proposition 5.4, $1 - \beta$ is not a fpc set. So, β is not a fpo set. $\square$

Proposition 5.6. *If the $\text{fp}\sigma$ -nd sets (α_k) 's are fpc sets in a fpVs (X, T) , then $\text{pcl}(\text{pint}(\bigwedge_{k=1}^N(\alpha_k))) = 1$.*

Proof. Let (α_k) 's denote the fpd and $\text{fp}G_\delta$ -sets. Then by the Theorem 2.26, $(1 - \alpha_k)$'s are $\text{fp}\sigma$ -nd sets. By hypothesis, the $\text{fp}\sigma$ -nd sets $(1 - \alpha_k)$'s are fpc sets and hence (α_k) 's are fpo sets. This implies that $\bigwedge_{k=1}^N (\alpha_k)$ is also a fpo set. Now $\text{pint}(\bigwedge_{k=1}^N (\alpha_k)) = \bigwedge_{k=1}^N (\alpha_k)$. Then $\text{pcl}(\text{pint}(\bigwedge_{k=1}^N (\alpha_k))) = \text{pcl}(\bigwedge_{k=1}^N (\alpha_k)) \longrightarrow (1)$. Since (X, T) is a fpVs, $\text{pcl}(\bigwedge_{k=1}^N (\alpha_k)) = 1$, where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets. From (1), $\text{pcl}(\text{pint}(\bigwedge_{k=1}^N (\alpha_k))) = 1$. $\square$

Proposition 5.7. *Let (α_k) 's be $\text{fp}\sigma$ -nd sets in a fpVs (X, T) . Then $\text{pint}(\bigvee_{k=1}^N (\alpha_k)) = 0$.*

Proof. Let (α_k) 's denote $\text{fp}\sigma$ -nd sets. Our aim is to prove that $\text{pint}(\bigvee_{k=1}^N (\alpha_k)) = 0$. Assume the contrary that $\text{pint}(\bigvee_{k=1}^N (\alpha_k)) \neq 0$. Then $1 - \text{pint}(\bigvee_{k=1}^N (\alpha_k)) \neq 1$. This implies that $\text{pcl}(\bigwedge_{k=1}^N (1 - \alpha_k)) \neq 1$. Since (α_k) 's are $\text{fp}\sigma$ -nd sets and by the Theorem 2.26, $(1 - \alpha_k)$'s are fpd and $\text{fp}G_\delta$ -sets. Then $\text{pcl}(\bigwedge_{k=1}^N (1 - \alpha_k)) \neq 1$, for the fpd and $\text{fp}G_\delta$ -sets $(1 - \alpha_k)$'s. Thus (X, T) is not a fpVs, a contradiction to the hypothesis that (X, T) is a fpVs. Therefore our assumption is wrong. Hence we must have $\text{pint}(\bigvee_{k=1}^N (\alpha_k)) = 0$. $\square$

Corollary 5.8. *If (α_k) 's are fpnd and $\text{fp}F_\sigma$ -sets in a fpVs, then $\text{pint}(\bigvee_{k=1}^N (\alpha_k)) = 0$.*

Proof. Let (α_k) 's denote fpnd and $\text{fp}F_\sigma$ -sets. Since (α_k) 's are fpnd sets, $\text{pintpcl}(\alpha_k) = 0$. But $\text{pint}(\alpha_k) \leq \text{pintpcl}(\alpha_k) = 0$ implies that $\text{pint}(\alpha_k) \leq 0$. That is, $\text{pint}(\alpha_k) = 0$. Hence (α_k) 's are $\text{fp}F_\sigma$ -sets with $\text{pint}(\alpha_k) = 0$. Then (α_k) 's are $\text{fp}\sigma$ -nd sets. Thus by the Proposition 5.7, $\text{pint}(\bigvee_{k=1}^N (\alpha_k)) = 0$. $\square$

Remark 5.9. In view of the Proposition 5.1 and the Corollary 5.8, one will have the following result:

“A fts is a fpVs if and only if $\text{pint}(\bigvee_{k=1}^N (\alpha_k)) = 0$, where (α_k) 's are fpnd and $\text{fp}F_\sigma$ -sets”.

Proposition 5.10. *If $\text{pint}(\bigvee_{k=1}^N (1 - \alpha_k)) = 0$, where (α_k) 's are $\text{fp}G_\delta$ -sets in (X, T) , then (X, T) is a fpVs.*

Proof. Let (α_k) 's denote the $\text{fp}G_\delta$ -sets such that $\text{pint}(\bigvee_{k=1}^N (1 - \alpha_k)) = 0$. But $\bigvee_{k=1}^N (\text{pint}(1 - \alpha_k)) \leq \text{pint}(\bigvee_{k=1}^N (1 - \alpha_k))$. Then $\bigvee_{k=1}^N (\text{pint}(1 - \alpha_k)) \leq 0$. That is, $\bigvee_{k=1}^N (\text{pint}(1 - \alpha_k)) = 0$. This implies that $\text{pint}(1 - \alpha_k) = 0$, ($i = 1$ to N). Thus $1 - \text{pcl}(\alpha_k) = 0$ and hence $\text{pcl}(\alpha_k) = 1$. Therefore, (α_k) 's are fpd sets. Since $\text{pint}(\bigvee_{k=1}^N (1 - \alpha_k)) = 0$, $1 - (\text{pcl}(\bigwedge_{k=1}^N (\alpha_k))) = 0$. Thus $\text{pcl}(\bigwedge_{k=1}^N (\alpha_k)) = 1$, where (α_k) 's are fpd and $\text{fp}G_\delta$ -sets. So, (X, T) is a fpVs. $\square$

Proposition 5.11. *If $pcl(\bigwedge_{k=1}^N(\alpha_k)) = 1$, where (α_k) 's are fpG_δ -sets in (X, T) , then (X, T) is a $fpVs$.*

Proof. Let (α_k) 's denote fpG_δ -sets such that $pcl(\bigwedge_{k=1}^N(\alpha_k)) = 1$. Then $1 - pcl(\bigwedge_{k=1}^N(\alpha_k)) = 0$. This implies that $pint(\bigvee_{k=1}^N(1 - \alpha_k)) = 0$. Then by the Proposition 5.10, (X, T) is a $fpVs$. $\square$

Proposition 5.12. *If (X, T) is a $fpVs$, then there exists a fpF_σ -set β in (X, T) such that $pint(\beta) \neq 0$.*

Proof. Let $\alpha = \bigwedge_{k=1}^N(\alpha_k)$, where (α_k) 's are fpd and fpG_δ sets. Since (X, T) is a $fpVs$, $pcl(\alpha) = pcl(\bigwedge_{k=1}^N(\alpha_k)) = 1$. Then $pcl(\alpha) = 1 \longrightarrow (1)$. Now $1 - pint(\alpha_k) = pcl(1 - \alpha_k)$ is a fpc set. Let $\beta = \bigvee_{k=1}^\infty(\beta_k)$, where (β_k) 's are fpc sets in which the first N fpc sets as $1 - pint(\alpha_k)$. Then β is a fpF_σ -set. But $\bigvee_{k=1}^N(1 - pint(\alpha_k)) \leq \bigvee_{k=1}^\infty(\beta_k)$. Then $1 - \bigwedge_{k=1}^N(pint(\alpha_k)) \leq \bigvee_{k=1}^\infty(\beta_k)$. Now $1 - \bigwedge_{k=1}^N(\alpha_k) < 1 - \bigwedge_{k=1}^N(pint(\alpha_k)) \leq \bigvee_{k=1}^\infty(\beta_k)$. Then $1 - \alpha < 1 - \bigwedge_{k=1}^N(pint(\alpha_k)) < \beta$. That is, $1 - \alpha < \beta$. This implies that $pint(1 - \alpha) < pint(\beta)$. Then $1 - pcl(\alpha) < pint(\beta)$. From (1), $1 - 1 < pint(\beta)$ and hence $pint(\beta) > 0$. That is, $pint(\beta) \neq 0$. $\square$

Proposition 5.13. *If each $fp\sigma$ -fc set is a $fp\sigma$ -nd set in (X, T) , then (X, T) is a $fpVs$.*

Proof. Let (α_k) 's are fpd and fpG_δ -sets. Then by the Theorem 2.26, $(1 - \alpha_k)$'s are $fp\sigma$ -nd sets. This will imply that $\bigvee_{k=1}^\infty(1 - \alpha_k)$ is a $fp\sigma$ -fs set. By hypothesis, $\bigvee_{k=1}^\infty(1 - \alpha_k)$ is a $fp\sigma$ -nd set. This implies that $\bigvee_{k=1}^\infty(1 - \alpha_k)$ is a fpF_σ -set such that $pint(\bigvee_{k=1}^\infty(1 - \alpha_k)) = 0$. Then $1 - pcl(\bigwedge_{k=1}^\infty(\alpha_k)) = 0$. Hence $pcl(\bigwedge_{k=1}^\infty(\alpha_k)) = 1$. But $pcl[\bigwedge_{k=1}^\infty(\alpha_k)] \leq pcl(\bigwedge_{k=1}^N(\alpha_k))$. Then $1 \leq pcl(\bigwedge_{k=1}^N(\alpha_k))$. That is, $pcl(\bigwedge_{k=1}^N(\alpha_k)) = 1$, where (α_k) 's are fpd and fpG_δ -sets. So, (X, T) is a $fpVs$. $\square$

Proposition 5.14. *If each fpG_δ -set has a fpd interior in (X, T) , then (X, T) is a $fpVs$.*

Proof. Let (α_k) 's denote the fpd and fpG_δ -sets. Then $\alpha = \bigwedge_{k=1}^N(\alpha_k)$ is a fpG_δ -set. By hypothesis, α has a fpd interior and hence $pcl\ pint(\alpha) = 1$. Now $pint(\alpha) \leq \alpha$, implies that $pint(\alpha) \leq \bigwedge_{k=1}^N(\alpha_k)$. Then $pcl\ pint(\alpha) \leq pcl(\bigwedge_{k=1}^N(\alpha_k))$. This implies that $1 \leq pcl(\bigwedge_{k=1}^N(\alpha_k))$. Thus $pcl(\bigwedge_{k=1}^N(\alpha_k)) \geq 1$. That is, $pcl(\bigwedge_{k=1}^N(\alpha_k)) = 1$, where (α_k) 's are fpd and fpG_δ -sets. So, (X, T) is a $fpVs$. $\square$

Proposition 5.15. *If (X, T) is a fVs , then (X, T) is not a $fpVs$.*

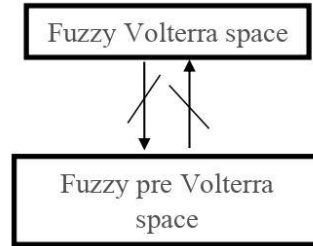
Proof. Let (X, T) denotes a fVs. Then $cl(\bigwedge_{k=1}^N(\alpha_k)) = 1$, where (α_k) 's are fd and fG_δ -sets. Since (α_k) 's are fG_δ -sets and by the Proposition 3.4, (α_k) 's are fpG_δ -sets. But by the Remark 3.3, the fd sets (α_k) 's are not the fpd sets. So, (X, T) is not a fpVs. $\square$

Proposition 5.16. *If (X, T) is a fpVs, then (X, T) is not a fVs.*

Proof. Let (X, T) denotes a fpVs. Then $pcl(\bigwedge_{k=1}^N(\alpha_k)) = 1$, where (α_k) 's are fpd and fpG_δ -sets. Since (α_k) 's are fpd sets and by the Proposition 3.2, (α_k) 's are fd sets. But by the Remark 3.8, the fpG_δ -sets (α_k) 's are not the fG_δ -sets. So, (X, T) is not a fVs. $\square$

Remark 5.17. From the above Propositions 5.15 and 5.16, one can conclude that fuzzy Volterra spaces and fuzzy pre Volterra spaces are independent.

Diagrammatically,



6. CONCLUSIONS AND FUTURE WORK

The new class of spaces called the fuzzy pre Volterra spaces with the help of fuzzy pre dense and fuzzy pre G_δ -sets have been introduced and studied in this paper. Examples were given to illustrate the concepts and by using these concepts, several interesting characterizations of fuzzy pre Volterra spaces were also explored in this paper. In the future, we plan to investigate the intriguing properties of fuzzy pre Volterra spaces and explore their interrelations with other fuzzy topological spaces. This study aims to offer deeper insights into the structural and useful aspects of these spaces, contributing to the broader understanding of fuzzy topology. Finally, we examined the notions of fuzzy Volterra spaces and fuzzy pre Volterra spaces are independent to each other.

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