

## WAVELET AMENABLE FRAMES IN SEPARABLE HILBERT SPACES

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**ABSTRACT.** We introduce wavelet amenable frames which are generated by the cyclic actions of a pair of special kind of operators on a single vector in separable Hilbert spaces and discuss some of its aspects. Frame operators of these frames are characterized. A necessary and sufficient condition for a pair of wavelet amenable frames to become canonical dual frames to each other is also obtained by observing an intertwining property of the frame operator.

### 1. INTRODUCTION

An interesting theme in frame theory is to construct frames by the cyclic actions of a pair of operators on a single reasonable vector in the given space. Weyl-Heisenberg frames (also known as Gabor frames) and wavelet frames are generated in  $L^2(\mathbb{R})$  in this manner where, translation, modulation and dilation operators are playing the role for the cyclic action. However, the main obstacle in general frame theory to carry forward the same idea is that the operators used for defining Gabor or wavelet frames can not be defined in general Hilbert spaces. We attempt to overcome this problem in the context of wavelet frames by bringing the dilation and translation operators defined on  $L^2(\mathbb{R})$  to act on the general separable Hilbert spaces through suitable bounded linear invertible transformations defined from  $L^2(\mathbb{R})$  into the Hilbert space concerned.

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## 2. PRELIMINARIES

Frames in Hilbert spaces are generalizations of orthonormal bases. The image of an orthonormal basis under an invertible operator is a frame known as a Riesz basis. Formally, a *frame* in a separable Hilbert space  $\mathcal{H}$  is a countable family  $\{u_k\}$  of vectors admitting some positive constants  $\alpha$  and  $\beta$  to satisfy the inequalities  $\alpha\|x\|^2 \leq \sum_k |\langle x, u_k \rangle|^2 \leq \beta\|x\|^2$  for every  $x \in \mathcal{H}$ .

The *frame operator* of a frame  $\{u_k\}$  in a Hilbert space  $\mathcal{H}$  is the bounded linear, positive, invertible operator  $S$  defined by  $Sx = \sum_k \langle x, u_k \rangle u_k$ ,  $x \in \mathcal{H}$ , where the series converges unconditionally in  $\mathcal{H}$ . For a given frame  $\{u_k\}$ , its image  $\{S^{-1}(u_k)\}$  under  $S^{-1}$  is also a frame in  $\mathcal{H}$ , known as the canonical dual frame of  $\{u_k\}$ .

A set of functions in  $L^2(\mathbb{R})$  of the form  $\{D_{a^k}T_{mb}\psi : k, m \in \mathbb{Z}\}$  formed by *dilates* and *translates* of a  $\psi \in L^2(\mathbb{R})$ , where  $a > 1, b > 0$  is called a *Wavelet system*. If a wavelet system becomes a frame, it is called a *wavelet frame*. Here the function  $\psi$  is known as the *generator* of the frame.

One of the main problem with the wavelet frame operator is that it need not commute with the involved translation operators  $T_{mb}$  of the wavelet frame concerned, so that the canonical dual frame need not have the wavelet frame structure (see [1], [4]).

More preliminary details on the topic of discussion of this article can be found in [1], [2], [8] and [9].

## 3. WAVELET AMENABLE FRAMES

Let  $\mathcal{H}$  be a separable Hilbert space and  $A : L^2(\mathbb{R}) \mapsto \mathcal{H}$  be an invertible bounded linear transformation. Then  $A$  maps every wavelet frame  $\mathcal{W} = \{D_{a^k}T_{mb}\psi : k, m \in \mathbb{Z}\}$  from  $L^2(\mathbb{R})$  to a frame  $A(\mathcal{W})$  in  $\mathcal{H}$ . In fact, the image of a frame under every surjective bounded linear transformation between two separable Hilbert spaces is again a frame (see [10]). The image frame  $A(\mathcal{W})$  has an interesting structure in  $\mathcal{H}$  whenever  $A : L^2(\mathbb{R}) \mapsto \mathcal{H}$  is invertible. We consider the map  $A$  to be invertible rather than unitary since non unitary approaches are also significant in many contexts like quantum field theory (see [11]). To observe and analyse this, we need the following notations and definition:

Given two invertible bounded linear transformations  $A, B : L^2(\mathbb{R}) \mapsto \mathcal{H}$  and  $k, m \in \mathbb{Z}$ , let  $D_{a^k}^A = AD_{a^k}A^{-1}$  and  $T_{mb}^B = BT_{mb}B^{-1}$ . Then  $D_{a^k}^A = (D_a^A)^k$  and  $T_{mb}^B = (T_b^B)^m$  for all  $k, m \in \mathbb{Z}$ .

**Definition 3.1.** Let  $\mathcal{K}$  and  $\mathcal{H}$  be separable Hilbert spaces over  $\mathbb{C}$ ,  $K \in \mathcal{B}(\mathcal{K})$  and  $H \in \mathcal{B}(\mathcal{H})$ . An operator  $\Psi \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  is said to be intertwining  $K$  and  $H$  locally at some  $\psi \in \mathcal{K}$  if  $\Psi(K\psi) = H(\Psi\psi)$ .

In the case when  $\mathcal{K} = \mathcal{H}$ ,  $\Psi \in \mathcal{B}(\mathcal{H})$  is said to be commuting with  $H$  locally at some  $\psi \in \mathcal{H}$  if  $\Psi(H\psi) = H(\Psi\psi)$  (see [7]).

**Proposition 3.2.** Let  $\mathcal{H}$  be a separable Hilbert space,  $A, B : L^2(\mathbb{R}) \mapsto \mathcal{H}$  be two invertible bounded linear maps and  $\mathcal{W} = \{D_{a^k}T_{mb}\psi : k, m \in \mathbb{Z}\}$  be a wavelet system in  $L^2(\mathbb{R})$ . Then  $A(\mathcal{W}) = \{D_{a^k}T_{mb}^B(A(\psi)) : k, m \in \mathbb{Z}\}$  if and only if  $A^{-1}B$  commutes locally with the family  $\{T_{mb}\}_{m \in \mathbb{Z}}$  at  $B^{-1}(A(\psi))$ .

*Proof.* For a wavelet system  $\mathcal{W} = \{D_{a^k}T_{mb}\psi : k, m \in \mathbb{Z}\}$  in  $L^2(\mathbb{R})$ , we have

$$\begin{aligned} A(\mathcal{W}) &= A(\{D_{a^k}T_{mb}\psi : k, m \in \mathbb{Z}\}) = \{AD_{a^k}T_{mb}\psi : k, m \in \mathbb{Z}\} \\ &= \{AD_{a^k}T_{mb}A^{-1}h : k, m \in \mathbb{Z}\} \text{ where } A(\psi) = h \in \mathcal{H}. \end{aligned}$$

Hence  $A(\mathcal{W}) = \{AD_{a^k}T_{mb}A^{-1}BB^{-1}h : k, m \in \mathbb{Z}\}$   
 $= \{AD_{a^k}A^{-1}BT_{mb}B^{-1}h : k, m \in \mathbb{Z}\}$  if  $A^{-1}B$  commutes locally with the family  $\{T_{mb}\}_{m \in \mathbb{Z}}$  at  $B^{-1}h = B^{-1}A(\psi)$ .

Thus  $A(\mathcal{W}) = \{D_{a^k}T_{mb}^Bh : k, m \in \mathbb{Z}\}$  in this case.

Conversely, if  $A(\mathcal{W}) = \{D_{a^k}T_{mb}^Bh : k, m \in \mathbb{Z}\}$ , then for all  $k, m \in \mathbb{Z}$ ,

$$AD_{a^k}T_{mb}\psi = D_{a^k}T_{mb}^Bh = AD_{a^k}A^{-1}BT_{mb}B^{-1}h$$

Since  $A(\psi) = h$ , this gives  $AD_{a^k}T_{mb}A^{-1}h = AD_{a^k}A^{-1}BT_{mb}B^{-1}h$ .

Then  $T_{mb}A^{-1}h = A^{-1}BT_{mb}B^{-1}h$  since  $AD_{a^k}$  is invertible.

Hence  $T_{mb}A^{-1}BB^{-1}h = A^{-1}BT_{mb}B^{-1}h$  for all  $m \in \mathbb{Z}$ , as required.  $\square$

In the proof above, the presence of the operator  $A^{-1}B$  between the operators  $D_{a^k}$  and  $T_{mb}$  is very delicate in the aspects of wavelets in  $L^2(\mathbb{R})$ , as discussed in [6]. We proceed with the following definition.

**Definition 3.3.** A wavelet amenable frame in a separable Hilbert space  $\mathcal{H}$  is a frame of the form  $\{D_{a^k}T_{mb}^Bh : k, m \in \mathbb{Z}\} = \{(D_a^A)^k(T_b^B)^mh : k, m \in \mathbb{Z}\}$  where  $a > 1$ ,  $b > 0$  and  $A, B : L^2(\mathbb{R}) \mapsto \mathcal{H}$  are invertible bounded linear maps.

**Proposition 3.4.** *Every separable Hilbert space contains wavelet amenable frames.*

*Proof.* For every pair  $(a, b)$ ,  $a > 1$ ,  $b > 0$ , of frame parameters, there is always a wavelet orthonormal basis  $\mathcal{W}_0 = \{D_{a^k}T_{mb}\psi_0\}_{k,m \in \mathbb{Z}}$  in  $L^2(\mathbb{R})$  (see [3]). For the same pair  $(a, b)$ , by Theorem.4 in [5], there are over complete non-tight wavelet frames, all with canonical duals of the same wavelet frame structure. The frame operator  $S$  of such a frame is invertible, non unitary and commutes with  $D_{a^k}$  and  $T_{mb}$  for all  $k, m \in \mathbb{Z}$ .

Let  $\mathcal{H}$  be a separable Hilbert space. Choose an orthonormal basis of the form  $\{e_{k,m} : k, m \in \mathbb{Z}\}$  in  $\mathcal{H}$ . Define  $U(f) = \sum_{k,m \in \mathbb{Z}} \langle f, D_{a^k}T_{mb}\psi_0 \rangle e_{k,m}$  for all  $f \in L^2(\mathbb{R})$ . Then  $U : L^2(\mathbb{R}) \mapsto \mathcal{H}$  is unitary. Let  $A = US$  and  $B = US^{-1}$ . Then  $A, B : L^2(\mathbb{R}) \mapsto \mathcal{H}$  are invertible, (but non unitary).

Since  $A^{-1}B = (S^{-1})^2$ , it commutes with  $T_{mb}$  for all  $m \in \mathbb{Z}$ . Thus, by Proposition 3.2,  $A(\mathcal{W}_0) = \{D_{a^k}T_{mb}^B A(\psi_0) : k, m \in \mathbb{Z}\}$ . and this image is a frame since  $A$  is surjective. Hence  $\{D_{a^k}T_{mb}^B A(\psi_0) : k, m \in \mathbb{Z}\}$  is a wavelet amenable frame in  $\mathcal{H}$ . In fact, it is a wavelet amenable Riesz basis in  $\mathcal{H}$ .  $\square$

At the end of this section, we provide a necessary and sufficient condition for a wavelet amenable frame to have the wavelet amenable frame structure for its canonical dual frame.

Analogous to the characterisation of the wavelet frame operator appeared in [7], a characterisation of the frame operators of wavelet amenable frames is presented now.

**Theorem 3.5.** *An operator  $S \in \mathcal{B}(\mathcal{H})$  becomes the frame operator of a frame  $\{D_{a^k}T_{mb}^B h\}_{k,m \in \mathbb{Z}}$  in  $\mathcal{H}$  if and only if  $S = \Psi\Psi^*$  where  $\Psi \in \mathcal{B}(\mathcal{H})$  is surjective and  $\Psi U : L^2(\mathbb{R}) \mapsto \mathcal{H}$  intertwines locally the families  $\{D_{a^k}T_{mb}\}_{k,m \in \mathbb{Z}}$  and  $\{D_{a^k}T_{mb}^B\}_{k,m \in \mathbb{Z}}$  at the generator of every wavelet basis  $\{D_{a^k}T_{mb}\psi_0\}_{k,m \in \mathbb{Z}}$  for every unitary transformation  $U : L^2(\mathbb{R}) \mapsto \mathcal{H}$ .*

*Proof.* Let  $S_{A,B}$  be the frame operator of a frame  $\{D_{a^k}T_{mb}^B h\}_{k,m \in \mathbb{Z}}$  in  $\mathcal{H}$ . If  $\mathcal{W}_0 = \{D_{a^k}T_{mb}\psi_0\}_{k,m \in \mathbb{Z}}$  is a wavelet orthonormal basis in  $L^2(\mathbb{R})$  with the same frame parameters  $a > 1$ ,  $b > 0$ , then for any unitary transformation  $U : L^2(\mathbb{R}) \mapsto \mathcal{H}$ ,  $U(\mathcal{W}_0)$  is an orthonormal basis for  $\mathcal{H}$ . Let  $h_0 = U(\psi_0)$ .

Define  $\Psi(x) = \sum_{k,m \in \mathbb{Z}} \langle x, UD_{a^k}T_{mb}\psi_0 \rangle D_{a^k}T_{mb}^B h$  for all  $x \in \mathcal{H}$ .

Then  $\Psi \in \mathcal{B}(\mathcal{H})$  maps  $U(\mathcal{W}_0)$  to the frame  $\{D_{a^k}T_{mb}^B h\}_{k,m \in \mathbb{Z}}$ .

Since  $\Psi(UD_{a^k}T_{mb}\psi_0) = D_{a^k}^AT_{mb}^Bh$  for all  $k, m \in \mathbb{Z}$  we obtain  $\Psi(h_0) = h$  and  $\Psi(UD_{a^k}T_{mb}\psi_0) = D_{a^k}^AT_{mb}^B\Psi(h_0)$  for all  $k, m \in \mathbb{Z}$ . Hence for all  $k, m \in \mathbb{Z}$ ,  $(\Psi U)D_{a^k}T_{mb}\psi_0 = D_{a^k}^AT_{mb}^B(\Psi U)(\psi_0)$ .

Thus the operator  $\Psi U : L^2(\mathbb{R}) \mapsto \mathcal{H}$  intertwines locally the families  $\{D_{a^k}T_{mb}\}_{k,m \in \mathbb{Z}}$  and  $\{D_{a^k}^AT_{mb}^B\}_{k,m \in \mathbb{Z}}$  at  $\psi_0$ .

Now

$$\begin{aligned} \Psi\Psi^*(x) &= \sum_{k,m \in \mathbb{Z}} \langle \Psi^*(x), UD_{a^k}T_{mb}\psi_0 \rangle D_{a^k}^AT_{mb}^Bh \\ &= \sum_{k,m \in \mathbb{Z}} \langle x, \Psi UD_{a^k}T_{mb}\psi_0 \rangle D_{a^k}^AT_{mb}^Bh \\ &= \sum_{k,m \in \mathbb{Z}} \langle x, D_{a^k}^AT_{mb}^Bh \rangle D_{a^k}^AT_{mb}^Bh \\ &= S_{A,B}(x) \text{ for all } x \in \mathcal{H}. \end{aligned}$$

Since  $S_{A,B}$  is invertible, for each  $h \in \mathcal{H}$ , there is  $h' \in \mathcal{H}$  with  $S_{A,B}(h') = h$ . Hence  $\Psi\Psi^*(h') = h$  so that  $\Psi(x) = h$  where  $x = \Psi^*(h')$ . Thus  $\Psi$  is surjective.

Conversely, assume that  $\Psi \in \mathcal{B}(\mathcal{H})$  is surjective and  $\Psi U$  intertwines the families  $\{D_{a^k}T_{mb}\}_{k,m \in \mathbb{Z}}$  and  $\{D_{a^k}^AT_{mb}^B\}_{k,m \in \mathbb{Z}}$  locally at the generator  $\psi_0$  of a wavelet basis  $\{D_{a^k}T_{mb}\psi_0\}_{k,m \in \mathbb{Z}}$  where  $U : L^2(\mathbb{R}) \mapsto \mathcal{H}$  is a unitary map.

Then  $\Psi U(\{D_{a^k}T_{mb}\psi_0\}_{k,m \in \mathbb{Z}}) = D_{a^k}^AT_{mb}^B(\Psi U)(\psi_0)$  is a frame in  $\mathcal{H}$  since  $\Psi U$  is surjective and  $\{D_{a^k}T_{mb}\psi_0\}_{k,m \in \mathbb{Z}}$  is an orthonormal basis. The frame operator of this frame is  $\Psi U(\Psi U)^* = \Psi U U^* \Psi^* = \Psi \Psi^*$ .  $\square$

Let  $A, B : L^2(\mathbb{R}) \mapsto \mathcal{H}$  be two invertible bounded linear maps. Then  $(A^*)^{-1}, (B^*)^{-1} : L^2(\mathbb{R}) \mapsto \mathcal{H}$  are also invertible bounded linear maps. Let  $(A^*)^{-1} = C$  and  $(B^*)^{-1} = D$ . Then a given wavelet frame  $\{D_{a^k}T_{mb}\psi\}_{k,m \in \mathbb{Z}}$  yields two wavelet amenable frames  $\{D_{a^k}^AT_{mb}^Bh_1\}_{k,m \in \mathbb{Z}}$  and  $\{D_{a^k}^CT_{mb}^Dh_2\}_{k,m \in \mathbb{Z}}$  respectively in  $\mathcal{H}$ . If  $\{D_{a^k}T_{mb}\psi\}_{k,m \in \mathbb{Z}}$  is a wavelet basis, these frames are dual frames to each other.

**Theorem 3.6.** *Let  $A, B, C, D : L^2(\mathbb{R}) \mapsto \mathcal{H}$  be invertible bounded linear maps. Then the frames  $\{D_{a^k}^AT_{mb}^Bh_1\}_{k,m \in \mathbb{Z}}$  and  $\{D_{a^k}^CT_{mb}^Dh_2\}_{k,m \in \mathbb{Z}}$  are dual frames to each other if and only if  $ASC^* = I_{\mathcal{H}} (= CSA^*)$ , where  $S$  is the frame operator of the wavelet frame  $\{D_{a^k}T_{mb}\psi\}_{k,m \in \mathbb{Z}}$  in  $L^2(\mathbb{R})$ .*

*Proof.* Let  $S$  be the frame operator of the wavelet frame  $\{D_{a^k}T_{mb}\psi\}_{k,m \in \mathbb{Z}}$  in  $L^2(\mathbb{R})$ . Then for all  $x \in \mathcal{H}$ ,

$$\begin{aligned}
& \sum_{k,m \in \mathbb{Z}} \langle x, D_{a^k}^C T_{mb}^D h_2 \rangle D_{a^k}^A T_{mb}^B h_1 \\
&= \sum_{k,m \in \mathbb{Z}} \langle x, C(D_{a^k} T_{mb} \psi) \rangle A(D_{a^k} T_{mb} \psi) \\
&= A \left( \sum_{k,m \in \mathbb{Z}} \langle C^*(x), D_{a^k} T_{mb} \psi \rangle D_{a^k} T_{mb} \psi \right) \\
&= ASC^*(x), \\
&= x \text{ if and only if } ASC^* = I_{\mathcal{H}}.
\end{aligned}$$

Similarly,  $\sum_{k,m \in \mathbb{Z}} \langle x, D_{a^k}^A T_{mb}^B h_1 \rangle D_{a^k}^C T_{mb}^D h_2 = x$  for all  $x \in \mathcal{H}$  if and only if  $CSA^* = I_{\mathcal{H}}$ .  $\square$

**Corollary 3.7.** *Let  $A, B : L^2(\mathbb{R}) \mapsto \mathcal{H}$  be two invertible bounded linear maps. Let  $C = (A^*)^{-1}$  and  $D = (B^*)^{-1}$ . If  $\{D_{a^k} T_{mb} \psi\}_{k,m \in \mathbb{Z}}$  is an orthonormal basis in  $L^2(\mathbb{R})$ , then the frames  $\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}$  and  $\{D_{a^k}^C T_{mb}^D h_2\}_{k,m \in \mathbb{Z}}$  are dual frames to each other.*

*Proof.* If  $\{D_{a^k} T_{mb} \psi\}_{k,m \in \mathbb{Z}}$  is an orthonormal basis in  $L^2(\mathbb{R})$ , then  $S = I$  so that the claim follows from Theorem 3.6 above.  $\square$

**Theorem 3.8.** *The canonical dual frame of a frame  $\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}$  in  $\mathcal{H}$  will be of the form  $\{D_{a^k}^C T_{mb}^D h_2\}_{k,m \in \mathbb{Z}}$  where  $C = (A^*)^{-1}$ ,  $D = S_{A,B}^{-1} B$  and  $S_{A,B}^{-1} h_1 = h_2$ .*

*Proof.* Let  $A, B : L^2(\mathbb{R}) \mapsto \mathcal{H}$  be two invertible bounded linear maps and  $S_{A,B}$  be the frame operator of the frame  $\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}$  in  $\mathcal{H}$ . Then

$$\begin{aligned}
S_{A,B}(D_{a^l}^A)^*(x) &= \sum_{k,m \in \mathbb{Z}} \langle (D_{a^l}^A)^*(x), D_{a^k}^A T_{mb}^B h_1 \rangle D_{a^k}^A T_{mb}^B h_1 \\
&= \sum_{k,m \in \mathbb{Z}} \langle x, D_{a^l}^A D_{a^k}^A T_{mb}^B h_1 \rangle D_{a^k}^A T_{mb}^B h_1 \\
&= \sum_{k,m \in \mathbb{Z}} \langle x, D_{a^{k+l}}^A T_{mb}^B h_1 \rangle D_{a^k}^A T_{mb}^B h_1 \\
&= \sum_{j,m \in \mathbb{Z}} \langle x, D_{a^j}^A T_{mb}^B h_1 \rangle D_{a^{j-l}}^A T_{mb}^B h_1 \\
&= D_{a^{-l}}^A \left( \sum_{j,m \in \mathbb{Z}} \langle x, D_{a^j}^A T_{mb}^B h_1 \rangle D_{a^j}^A T_{mb}^B h_1 \right) \\
&= D_{a^{-l}}^A S_{A,B}(x) \text{ for all } x \in \mathcal{H}.
\end{aligned}$$

Thus  $S_{A,B}(D_{a^l}^A)^* = D_{a^{-l}}^A S_{A,B}$  or  $S_{A,B}(D_{a^l}^A)^* = D_{a^l}^A S_{A,B}$  and hence the frame operator  $S_{A,B}$  of the frame  $\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}$  intertwines the operators  $D_{a^{-l}}^A$  with  $(D_{a^l}^A)^*$ ,  $l \in \mathbb{Z}$ . However,  $S_{A,B}$  need not have this property in terms of the operators  $T_{mb}^B$ ,  $m \in \mathbb{Z}$ .

Since  $S_{A,B}(D_{a-l}^A)^* = D_{a-l}^A S_{A,B} \Rightarrow S_{A,B}^{-1} D_{a-l}^A = (D_{a-l}^A)^* S_{A,B}^{-1}$  for all  $l \in \mathbb{Z}$ , the canonical dual frame of the frame  $\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}$  is given by

$$\begin{aligned}
 S_{A,B}^{-1}(\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}) &= S_{A,B}^{-1}(\{D_{a^k}^A B T_{mb} B^{-1} h_1\}_{k,m \in \mathbb{Z}}) \\
 &= \{S_{A,B}^{-1} D_{a^k}^A B T_{mb} B^{-1} S_{A,B} S_{A,B}^{-1} h_1\}_{k,m \in \mathbb{Z}} \\
 &= \{(D_{a^{-k}}^A)^* (S_{A,B}^{-1} B T_{mb} (S_{A,B}^{-1} B)^{-1}) (S_{A,B}^{-1} h_1)\}_{k,m \in \mathbb{Z}} \\
 &= \{(D_{a^{-k}}^A)^* T_{mb}^{S_{A,B}^{-1} B} S_{A,B}^{-1} h_1\}_{k,m \in \mathbb{Z}} \\
 &= \{(D_{a^{-k}}^A)^* T_{mb}^D h_2\}_{k,m \in \mathbb{Z}}, \text{ where } D = S_{A,B}^{-1} B \text{ and } S_{A,B}^{-1} h_1 = h_2 \\
 &= \{(A D_{a^{-k}} A^{-1})^* T_{mb}^D h_2\}_{k,m \in \mathbb{Z}} \\
 &= \{(A^{-1})^* D_{a^k} A^* T_{mb}^D h_2\}_{k,m \in \mathbb{Z}} \\
 &= \{D_{a^k}^C T_{mb}^D h_2\}_{k,m \in \mathbb{Z}} \text{ where } C = (A^*)^{-1}. \quad \square
 \end{aligned}$$

**Corollary 3.9.** *The canonical dual frame of a frame  $\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}$  in  $\mathcal{H}$  will be of the form  $\{(D_{a^{-k}}^A)^* T_{mb}^B h_2\}_{k,m \in \mathbb{Z}}$  if and only if  $S_{A,B}^{-1}$  commutes locally with  $T_{mb}^B$  for all  $m \in \mathbb{Z}$  at  $h_1 \in \mathcal{H}$ .*

*Proof.* By Theorem 3.8, the canonical dual frame of the frame  $\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}$  will be  $\{(D_{a^{-k}}^A)^* T_{mb}^C h_2\}_{k,m \in \mathbb{Z}}$ , where  $C = S_{A,B}^{-1} B$  and  $S_{A,B}^{-1} h_1 = h_2$ . Hence

$$S_{A,B}^{-1}(\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}) = \{(D_{a^{-k}}^A)^* (S_{A,B}^{-1} B) T_{mb} (S_{A,B}^{-1} B)^{-1} S_{A,B}^{-1} h_1\}_{k,m \in \mathbb{Z}}$$

so that

$$\begin{aligned}
 S_{A,B}^{-1}(\{D_{a^k}^A T_{mb}^B h_1\}_{k,m \in \mathbb{Z}}) &= \{(D_{a^{-k}}^A)^* B T_{mb} B^{-1} S_{A,B}^{-1} h_1\}_{k,m \in \mathbb{Z}} \\
 &= \{(D_{a^{-k}}^A)^* T_{mb}^B S_{A,B}^{-1} h_1\}_{k,m \in \mathbb{Z}}
 \end{aligned}$$

if and only if  $S_{A,B}^{-1}$  commutes locally with  $T_{mb}^B$  for all  $m \in \mathbb{Z}$  at  $h_1 \in \mathcal{H}$ .  $\square$

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