

COMPLEX FUZZY IDEALS IN BCK-ALGEBRAS

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ABSTRACT. The notions of complex fuzzy ideals in the sense of min-complex t -norm are introduced, and their relations and related properties are investigated. We explored the properties of sufficient and necessary conditions for complex fuzzy ideals.

1. INTRODUCTION

The notion of fuzzy sets was introduced by Zadeh [11]. The fuzzy set theory expanding the range of the membership function from $\{0, 1\}$ to the unit interval $[0, 1]$ is a mathematically analogous to the extension of the set of integers $\mathbb{Z}$ to the set of real numbers $\mathbb{R}$. The development of the number set did not end with real numbers. The introduction of real numbers was followed by their extension to the set of complex numbers $\mathbb{C}$. Hence it may be suggested that a further development of fuzzy set theory should be based on this extension. In the context of set theory, the result of such an extension is the complex fuzzy set, i.e., a fuzzy set characterized by a complex-valued membership function. Based on such background, Ramot et al. introduced complex fuzzy set in their paper [8]. For more details on complex fuzzy sets, we refer the readers to [1, 2, 9].

Ramot et al. developed the theory of complex fuzzy sets by introducing homogeneous between two complex fuzzy sets. Homogeneous has its advantages, but it has limitations and has the effect of overly lumping 'radius' and 'angle' on the complex

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plane. Efforts to overcome these limitations were made by Jun and Xin [6]. The basic idea of Jun and Xin's complex fuzzy set operation is as follows. The complex fuzzy set distinguishes between 'radius' and 'angle' that appear on the unit disc on the complex plane. A new operation was defined between two complex fuzzy sets by defining the operation with t-norm for 'radius' and extended t-norm for 'angle'.

In this study, following Jun and Xin's ideas, the notions of complex fuzzy ideals in the sense of min-complex t-norm are introduced, and their relations and related properties are investigated. We explored the properties of sufficient and necessary conditions for complex fuzzy ideals.

2. PRELIMINARIES

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [4] and [5]) and was extensively investigated by several researchers. We recall the definitions and basic results required in this paper. See the books [3, 7] for further information regarding BCK/BCI-algebras.

By a *BCI-algebra*, we mean a structure $(X, *, 0)$, where 0 is a special element and $*$ is a binary operation on X , that satisfies the following conditions:

- (I) $((a * b) * (a * c)) * (c * b) = 0$,
- (II) $(a * (a * b)) * b = 0$,
- (III) $a * a = 0$,
- (IV) $a * b = 0, b * a = 0 \Rightarrow a = b$,

for all $a, b, c \in X$. If a BCI-algebra $(X, *, 0)$ satisfies the following identity:

- (V) $(\forall a \in X) (0 * a = 0)$,

then $(X, *, 0)$ is called a *BCK-algebra*.

The order relation " $\leq_X$ " in a BCK/BCI-algebra $(X, *, 0)$ is defined as follows:

- (1) $(\forall a, b \in X) (a \leq_X b \Leftrightarrow a * b = 0)$.

Every BCK/BCI-algebra $(X, *, 0)$ satisfies the following conditions:

- (2) $a * 0 = a$,
- (3) $a \leq_X b \Rightarrow a * c \leq_X b * c, c * b \leq_X c * a$,
- (4) $(a * b) * c = (a * c) * b$,

for all $a, b, c \in X$.

Every BCK-algebra $(X, *, 0)$ satisfies:

$$(5) \quad (\forall x, a \in X)(x * a \leq_X x).$$

A nonempty subset S of X is called a *subalgebra* of a BCK/BCI-algebra $(X, *, 0)$ (see [7]) if $a * y \in S$ for all $a, y \in S$. A subset A of X is called an *ideal* of a BCK/BCI-algebra $(X, *, 0)$ (see [7]) if it satisfies:

$$(6) \quad 0 \in A,$$

$$(7) \quad (\forall a \in X)(\forall y \in A)(a * y \in A \Rightarrow a \in A).$$

Every ideal A of a BCK/BCI-algebra $(X, *, 0)$ satisfies the next assertion.

$$(8) \quad (\forall a, y \in X)(a \leq_X y, y \in A \Rightarrow a \in A).$$

A *fuzzy set* A is a pair (X, μ_A) where X is a nonempty set and $\mu_A : X \rightarrow [0, 1]$ a membership function, the value $\mu_A(x)$ is called the grade of membership of x in (X, μ_A) . It will be denoted by $A = \{(x, \mu_A(x)) | x \in X\}$.

A fuzzy set $A = \{(x, \mu_A(x)) | x \in X\}$ on a BCK-algebra $(X, *, 0)$ is called a *fuzzy subalgebra* of X (see [10]) if it satisfies:

$$(9) \quad (\forall x, y \in X)(\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}).$$

A fuzzy set $A = \{(x, \mu_A(x)) | x \in X\}$ on a BCK-algebra $(X, *, 0)$ is called a *fuzzy ideal* of X (see [10]) if it satisfies:

$$(10) \quad (\forall x \in X)(\mu_A(0) \geq \mu_A(x)),$$

$$(11) \quad (\forall x, y \in X)(\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}).$$

Ramot et al. [8] introduced the innovative complex fuzzy set.

Definition 2.1. [8] Let X be a nonempty set. A *complex fuzzy set* A on X is an object having the form $A = \{(x, \mu_A(x)) | x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in X$ a complex number $\mu_A(x)$ lies within the unit circle in the complex plane.

We shall assume that $\mu_A(x)$ will be represented by $r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, 2\pi]$. Note that by setting $\omega_A(x) = 0$ in the Definition 2.1, we return back to the fuzzy set.

Let $\mu_A(x)$ and $\mu_A(y)$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_A(x) \succeq \mu_A(y)$, we mean $\text{Re}(\mu_A(x)) \geq \text{Re}(\mu_A(y))$ and $\text{Im}(\mu_A(x)) \geq \text{Im}(\mu_A(y))$.

Jun and Xin [6] introduced the operation between two of complex fuzzy sets in the following way.

Let A and B be complex fuzzy sets on a nonempty set X with the complex valued membership functions μ_A and μ_B , respectively. We define

$$(12) \quad \mu_A(x) \odot \mu_B(y) = [r_A(x) \diamond r_B(y)] \cdot e^{i[\omega_A(x) \bar{\diamond} \omega_B(y)]}$$

for all $x, y \in X$, where $\diamond$ is a t-norm and $\bar{\diamond}$ is a function

$$\bar{\diamond} : [0, \pi] \times [0, \pi] \rightarrow [0, \pi]$$

satisfies the conditions associativity, commutativity, monotonicity and π is the identity element. We say that the function $\bar{\diamond}$ is an *extended t-norm*, and the operation $\odot$ is the *complex t-norm*.

In what follows, we consider a complex fuzzy set $A = \{(x, \mu_A(x)) | x \in X\}$ on a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. For any $\delta = r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$ and $\omega_A \in [0, \pi]$, the δ -level set of A is denoted by $[A]_\delta$ and is denoted to be the set

$$[A]_\delta := \{x \in X | \mu_A(x) \succeq \delta\}.$$

If, in the complex t-norm $\odot$, both the t-norm and extended t-norm are considered as “min”, it is denoted by $\odot_{\min}$ and is called the *min-complex t-norm*.

Definition 2.2. [6] A complex fuzzy set $A = \{(x, \mu_A(x)) | x \in X\}$ on a BCK-algebra X is called a *complex fuzzy subalgebra* of X if

$$(13) \quad \begin{cases} \operatorname{Re}(\mu_A(x * y)) \geq \operatorname{Re}(\mu_A(x) \odot \mu_A(y)), \\ \operatorname{Im}(\mu_A(x * y)) \geq \operatorname{Im}(\mu_A(x) \odot \mu_A(y)) \end{cases}$$

or, equivalently, $\mu_A(x * y) \succeq \mu_A(x) \odot \mu_A(y)$ for all $x, y \in X$.

Lemma 2.3. [6] Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy subalgebra of a BCK-algebra X . If we use the min-complex t-norm, then $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$.

3. COMPLEX FUZZY IDEALS IN BCK-ALGEBRAS

Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a fuzzy set on a BCK-algebra X . Then

$$A_\pi := \{(x, \delta_A(x)) | \delta_A(x) = \pi \mu_A(x), x \in X\}$$

is a fuzzy set with domain X and image $[0, \pi]$. We will call A_π the $[0, \pi]$ -fuzzy set.

Definition 3.1. A $[0, \pi]$ -fuzzy set A_π on a BCK-algebra $(X, *, 0)$ is called a $[0, \pi]$ -fuzzy subalgebra of X if it satisfies:

$$(14) \quad (\forall x, y \in X)(\delta_A(x * y) \geq \min\{\delta_A(x), \delta_A(y)\}).$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra([7]) with the table of “ $*$ ” given by the Figure 1.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Figure 1. Table of “ $*$ ”

Let A be a fuzzy set on X with the $[0, 1]$ valued membership function $\mu_A : X \rightarrow [0, 1]$ defined by

$$\mu_A(x) := \begin{cases} 0.6 & \text{if } x \in \{0, 2\}, \\ 0.2 & \text{otherwise,} \end{cases}$$

and let A_π be a $[0, \pi]$ -fuzzy set on X with the $[0, \pi]$ valued membership function $\delta_A : X \rightarrow [0, \pi]$ defined by $\delta_A(x) = \pi\mu_A(x)$ for all $x \in X$. Then $A_\pi = \{(x, \delta_A(x)) | x \in X\}$ is a $[0, \pi]$ -fuzzy subalgebra of X .

The following theorem gives the relation between complex fuzzy subalgebras and $[0, \pi]$ -fuzzy subalgebras.

Theorem 3.3. Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a fuzzy set of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. If we use the min-complex t -norm, then A is a complex fuzzy subalgebra of X if and only if $R := \{(x, r_A(x)) | r_A(x) \in [0, 1], x \in X\}$ is a fuzzy subalgebra of X and $R_\pi := \{(x, \omega_A(x)) | \omega_A(x) \in [0, \pi], x \in X\}$ is a $[0, \pi]$ -fuzzy subalgebra of X .

Proof. The proof is easily obtained, so it is omitted. $\square$

Let $A = \{(x, \mu_A(x)) | x \in X\}$ and $B = \{(x, \mu_B(x)) | x \in X\}$ be two complex fuzzy sets of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{i\omega_B(x)}$ with $i = \sqrt{-1}$, $r_A(x), r_B(x) \in [0, 1]$ and $\omega_A(x), \omega_B(x) \in [0, \pi]$. Then we introduce the complex fuzzy intersection of μ_A and μ_B of X as follows:

$$A \cap B := \{(x, \mu_{A \cap B}(x)) | x \in X\},$$

where $\mu_{A \cap B} = r_{A \cap B}(x) \cdot e^{i\omega_{A \cap B}(x)} = \min\{r_A(x), r_B(x)\} \cdot e^{i \min\{\omega_A(x), \omega_B(x)\}}$.

We have the following theorem.

Theorem 3.4. Let $\{A_j\}_{j \in \Lambda}$ be a collection of complex fuzzy subalgebras of a BCK-algebra X in which $\mu_{A_j}(x) = r_{A_j}(x) \cdot e^{i\omega_{A_j}(x)}$ with $i = \sqrt{-1}$, $r_{A_j}(x) \in [0, 1]$ and $\omega_{A_j}(x) \in [0, \pi]$. If we use the min-complex t-norm, then

$$\bigcap_{j \in \Lambda} A_j = \{(x, \mu_{\bigcap_{j \in \Lambda} A_j}(x)) | x \in X\}$$

is a complex fuzzy subalgebra of X .

Proof. The proof is easily obtained, so it is omitted. $\square$

Definition 3.5. A complex fuzzy set $A = \{(x, \mu_A(x)) | x \in X\}$ on a BCK-algebra X is called a *complex fuzzy ideal* of X if

$$(15) \quad (\forall x \in X)(\mu_A(0) \succeq \mu_A(x)),$$

$$(16) \quad (\forall x, y \in X)(\mu_A(x) \succeq \mu_A(x * y) \odot \mu_A(y)).$$

Example 3.6. Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra([7]) with the Hasse diagram of $(X, \leq_X)$ and the table of “ $*$ ” given by the Figure 2 ,

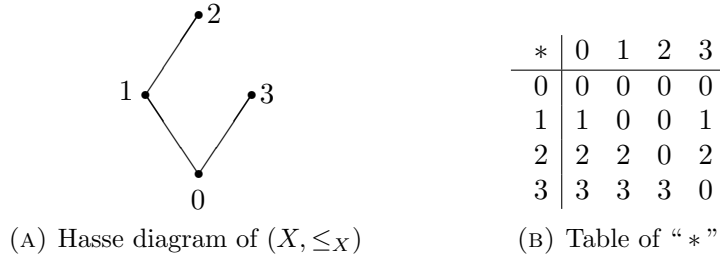


Figure 2. Hasse diagram and Table of X

Let A be a complex fuzzy set on X with the complex valued membership function μ_A defined by

$$\mu_A(x) := \begin{cases} 0.7e^{i\frac{2\pi}{7}} & \text{if } x = 0, \\ 0.5e^{i\frac{2\pi}{7}} & \text{if } x = 1, \\ 0.3e^{i\frac{2\pi}{7}} & \text{if } x = 2, \\ 0.1e^{i\frac{2\pi}{7}} & \text{if } x = 3. \end{cases}$$

If we use the min-complex t-norm, then $A = \{(x, \mu_A(x)) | x \in X\}$ is a complex fuzzy ideal of X .

Proposition 3.7. *Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy ideal of a BCK-algebra X . If we use the min-complex t-norm, then*

$$(17) \quad (\forall x, y \in X)(x \leq_X y \Rightarrow \mu_A(x) \succeq \mu_A(y)).$$

Proof. Let $x, y \in X$ be such that $x \leq_X y$. Then by Definition 3.5 that we have

$$\mu_A(x) \succeq \mu_A(x * y) \odot \mu_A(y) = \mu_A(0) \odot \mu_A(y) \succeq \mu_A(y),$$

ending the proof. $\square$

Theorem 3.8. *Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy ideal of a BCK-algebra X . If we use the min-complex t-norm, then A is a complex fuzzy subalgebra of X .*

Proof. Let $x, y \in X$. we have $x * y \leq_X x$ by (5), it follows from that Proposition 3.7 that $\mu_A(x * y) \succeq \mu_A(x)$. Hence we get

$$\mu_A(x * y) \succeq \mu_A(x) \succeq \mu_A(x * y) \odot \mu_A(y) \succeq \mu_A(x) \odot \mu_A(y).$$

Therefore A is a complex fuzzy subalgebra of X . $\square$

A complex fuzzy subalgebra of a BCK-algebra X may not be a complex fuzzy ideal of X as shown following Example.

Example 3.9. In the Example 3.6, let A be a complex fuzzy set on X with the complex valued membership function μ_A defined by

$$\mu_A(x) := \begin{cases} 0.8e^{i\frac{9\pi}{20}} & \text{if } x = 0, \\ 0.4e^{i\frac{2\pi}{5}} & \text{if } x = 1, \\ 0.6e^{i\frac{7\pi}{20}} & \text{if } x = 2, \\ 0.1e^{i\frac{3\pi}{10}} & \text{if } x = 3. \end{cases}$$

If we use the min-complex t-norm, then $A = \{(x, \mu_A(x)) | x \in X\}$ is a complex fuzzy subalgebra of X but not a complex fuzzy ideal of X , since $Re(\mu_A(1)) = 0.4\cos\frac{8\pi}{20} \not\geq 0.6\cos\frac{7\pi}{20} = Re(\mu_A(0) \odot \mu_A(2)) = Re(\mu_A(1 * 2) \odot \mu_A(2))$ and so $\mu_A(1) \not\succeq \mu_A(1 * 2) \odot \mu_A(2)$.

Definition 3.10. A $[0, \pi]$ -fuzzy set A_π on a BCK-algebra $(X, *, 0)$ is called a $[0, \pi]$ -fuzzy ideal of X if it satisfies:

$$(18) \quad (\forall x \in X)(\delta_A(0) \geq \delta_A(x)),$$

$$(19) \quad (\forall x, y \in X)(\delta_A(x) \geq \min\{\delta_A(x * y), \delta_A(y)\}).$$

Example 3.11. In Example 3.2, let A be a fuzzy set on X with the $[0, 1]$ valued membership function $\mu_A : X \rightarrow [0, 1]$ defined by

$$\mu_A(x) := \begin{cases} 0.7 & \text{if } x = 0, \\ 0.4 & \text{if } x \in \{1, 2\}, \\ 0.3 & \text{otherwise,} \end{cases}$$

and let A_π be a $[0, \pi]$ -fuzzy set on X with the $[0, \pi]$ valued membership function $\delta_A : X \rightarrow [0, \pi]$ defined by $\delta_A(x) = \pi\mu_A(x)$ for all $x \in X$. Then $A_\pi = \{(x, \delta_A(x)) | x \in X\}$ is a $[0, \pi]$ -fuzzy ideal of X .

The following theorem gives the relation between complex fuzzy ideals and $([0, \pi]$ -fuzzy ideals.

Theorem 3.12. Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a fuzzy set of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. If we use the min-complex t -norm, then A is a complex fuzzy ideal of X if and only if $R = \{(x, r_A(x)) | r_A(x) \in [0, 1], x \in X\}$ is a fuzzy ideal of X and $R_\pi = \{(x, \omega_A(x)) | \omega_A(x) \in [0, \pi], x \in X\}$ is a $[0, \pi]$ -fuzzy ideal of X .

Proof. The proof can be obtained without difficulty, so it is omitted. $\square$

Theorem 3.13. Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy ideal of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. If we use the min-complex t -norm, then the δ -level set $[A]_\delta$ of A is an ideal of X for all $\delta = r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$ and $\omega_A \in [0, \pi]$ and $[A]_\delta \neq \emptyset$.

Proof. By Theorem 3.8 and Lemma 2.3, we have $\mu_A(0) \succeq r_A \cdot e^{i\omega_A} = \delta$ and so $0 \in [A]_\delta$. Let $x, y \in X$ be such that $x * y \in [A]_\delta$ and $y \in [A]_\delta$. Then we obtain $\mu_A(x * y) \succeq \delta$ and $\mu_A(y) \succeq \delta$. Hence we get

$$\mu_A(x) \succeq \mu_A(x * y) \odot \mu_A(y) \succeq \delta,$$

and so $x \in [A]_\delta$. Therefore $[A]_\delta$ is an ideal of X . $\square$

Corollary 3.14. Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy ideal of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. If we use the min-complex t -norm, then the set

$$[X] := \{x \in X | \mu_A(x) = \mu_A(0)\}$$

is an ideal of X .

Proof. By Theorem 3.8 and Lemma 2.3, we have

$$[A]_{\mu_A(0)} = \{x \in X \mid \mu_A(x) \succeq \mu_A(0)\} = \{x \in X \mid \mu_A(x) = \mu_A(0)\} = [X].$$

It follows from Theorem 3.13 that $[X]$ is an ideal of X . $\square$

Theorem 3.15. *Let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex fuzzy set of a BCK-algebra X such that the δ -level set $[A]_\delta$ of A is an ideal of X for all $\delta = r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$, $\omega_A \in [0, \pi]$ and $[A]_\delta \neq \emptyset$. If we use the min-complex t -norm, then A is a complex fuzzy ideal of X .*

Proof. Suppose that the δ -level set $[A]_\delta$ of A is an ideal of X for all $\delta = r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$, $\omega_A \in [0, \pi]$ and $[A]_\delta \neq \emptyset$. Assume that there exists $x \in X$ such that $\mu_A(0) \not\succeq \mu_A(x)$. We take $\delta := r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$ and $\omega_A \in [0, \pi]$ such that

$$\mu_A(0) \prec \delta \preceq \mu_A(x).$$

Then $0 \notin [A]_\delta$. This is a contradiction by (15). Assume that there exist $x, y \in X$ such that $\mu_A(x) \prec \mu_A(x * y) \odot \mu_A(y)$. We take $\delta := r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$ and $\omega_A \in [0, \pi]$ such that

$$\mu_A(x) \prec \delta \preceq \mu_A(x * y) \odot \mu_A(y).$$

Then $x \notin [A]_\delta, x * y \in [A]_\delta$ and $y \in [A]_\delta$. This is a contradiction by (16). Therefore A is a complex fuzzy ideal of X . $\square$

Corollary 3.16. *For any subset B of a BCK-algebra X , let $A = \{(x, \mu_A(x)) \mid x \in X\}$ be a complex fuzzy set on X with the complex valued membership function μ_A defined by*

$$\mu_A(x) := \begin{cases} r_1 \cdot e^{i\omega_1} & \text{if } x \in B, \\ r_2 \cdot e^{i\omega_2} & \text{otherwise} \end{cases}$$

where $r_1 \cdot e^{i\omega_1} \succeq r_2 \cdot e^{i\omega_2}$, $r_1, r_2 \in [0, 1]$ and $\omega_1, \omega_2 \in [0, \pi]$. If B is an ideal of X and we use the min-complex t -norm, then $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex fuzzy ideal of X

Proof. Since $B = [A]_{r_1 \cdot e^{i\omega_1}}$ and B is an ideal of X , we get $B \neq \emptyset$. By Theorem 3.15, we have $A = \{(x, \mu_A(x)) \mid x \in X\}$ is a complex fuzzy ideal of X . $\square$

Theorem 3.17. *Let $\{A_j\}_{j \in \Lambda}$ be a collection of complex fuzzy ideals of a BCK-algebra X in which $\mu_{A_j}(x) = r_{A_j}(x) \cdot e^{i\omega_{A_j}(x)}$ with $i = \sqrt{-1}$, $r_{A_j}(x) \in [0, 1]$ and $\omega_{A_j}(x) \in [0, \pi]$. If we use the min-complex t -norm, then*

$$\bigcap_{j \in \Lambda} A_j = \{(x, \mu_{\bigcap_{j \in \Lambda} A_j}(x)) \mid x \in X\}$$

is a complex fuzzy ideal of X .

Proof. The proof is easily obtained, so it is omitted. $\square$

Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy ideal of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. Each ideals $[A]_\delta$ of X , $\delta = r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$ and $\omega_A \in [0, \pi]$, is called a *level ideal* of A , when $[A]_\delta \neq \emptyset$ and we use the min-complex t-norm.

Theorem 3.18. *Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy ideal of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. Then two level ideals $[A]_{\delta_1}$ and $[A]_{\delta_2}$ with $\delta_1 \prec \delta_2$ of A are equal if and only if there is no $x \in X$ such that $\delta_1 \preceq \mu_A(x) \prec \delta_2$.*

Proof. Suppose that $\delta_1 \prec \delta_2$ and $[A]_{\delta_1} = [A]_{\delta_2}$. If there exists $x \in X$ such that $\delta_1 \preceq \mu_A(x) \prec \delta_2$, then $[A]_{\delta_2}$ is a proper subset of $[A]_{\delta_1}$. This is a contradiction. Conversely, suppose that there is no $x \in X$ such that $\delta_1 \preceq \mu_A(x) \prec \delta_2$. Note that $\delta_1 \prec \delta_2$ implies $[A]_{\delta_2} \subseteq [A]_{\delta_1}$. If $x \in [A]_{\delta_1}$, then $\mu_A(x) \succeq \delta_1$, and so $\mu_A(x) \succeq \delta_2$. Hence $x \in [A]_{\delta_2}$, which says that $[A]_{\delta_1} \subseteq [A]_{\delta_2}$. Therefore $[A]_{\delta_1} = [A]_{\delta_2}$. $\square$

Remark 3.19. As a consequence of Theorem 3.18, the level ideals of complex fuzzy ideal $A = \{(x, \mu_A(x)) | x \in X\}$ of a BCK-algebra X which has a countable image form a chain. But $\mu_A(0) \succeq \mu_A(x)$ for all $x \in X$, and so $[A]_{\mu_A(0)}$ is the smallest level ideal of A , but not always $[A]_{\mu_A(0)} = \{0\}$ as shown in the following example.

Let B be a proper ideal of a BCK-algebra X and let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy set of X in Corollary 3.16. Then $Im(\mu_A) = \{r_1 \cdot e^{i\omega_1}, r_2 \cdot e^{i\omega_2}\}$, and two level ideals of A are $[A]_{r_2 \cdot e^{i\omega_2}} = X$ and $[A]_{r_1 \cdot e^{i\omega_1}} = B$. Thus we have $\mu_A(0) = r_1 \cdot e^{i\omega_1}$ but $[A]_{\mu_A(0)} = B \neq \{0\}$.

Thus we have a chain

$$[A]_{\mu_A(0)} \subseteq \cdots \subseteq [A]_{\delta_k} \subseteq \cdots \subseteq [A]_{\delta_1} \subseteq [A]_{\delta_0} = X,$$

where $\delta_0 \preceq \cdots \preceq \delta_k \preceq \cdots$ and $\mu_A(0) = \lim_{n \rightarrow \infty} \delta_n$.

Corollary 3.20. *Let $A = \{(x, \mu_A(x)) | x \in X\}$ be a complex fuzzy ideal of a BCK-algebra X in which $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ with $i = \sqrt{-1}$, $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, \pi]$. If $Im(\mu_A) = \{\delta_1, \delta_2, \dots, \delta_n\}$, where $\delta_1 \prec \delta_2 \prec \cdots \prec \delta_n$, then the family of ideals $[A]_{\delta_j}$ of A ($j = 1, 2, \dots, n$) constitutes all the level ideals of A .*

Proof. Let $\delta = r_A \cdot e^{i\omega_A}$ with $r_A \in [0, 1]$, $\omega_A \in [0, \pi]$ and $\delta \notin \text{Im}(\mu_A)$. If $\delta \prec \delta_1$, then $[A]_{\delta_1} \subseteq [A]_{\delta}$. Since $[A]_{\delta_1} = X$, we have $[A]_{\delta} = X$ and $[A]_{\delta_1} = [A]_{\delta}$. If $\delta_j \prec \delta \prec \delta_{j+1}$ ($1 \geq j \geq n-1$), then there is no $x \in X$ such that $\delta_1 \preceq \mu_A(x) \prec \delta_{j+1}$. Using Theorem 3.18, we obtain $[A]_{\delta} = [A]_{\delta_{j+1}}$. This shows that for any δ with $\delta \prec \mu_A(0)$, the level ideal $[A]_{\delta}$ is in $\{[A]_{\delta_j} | a \leq j \leq n\}$. $\square$

4. CONCLUSION

In this paper, we introduced the notion of complex fuzzy ideals in BCK-algebras based on the min-complex t-norm and investigated their fundamental properties and relationships with related algebraic structures. In particular, we established the connections between complex fuzzy ideals and ordinary fuzzy ideals as well as $[0, \pi]$ -fuzzy ideals by separating the radius and angle components of complex-valued membership functions.

We proved that every complex fuzzy ideal is a complex fuzzy subalgebra under the min-complex t-norm and characterized complex fuzzy ideals in terms of their corresponding fuzzy ideals and $[0, \pi]$ -fuzzy ideals. Moreover, we showed that the δ -level sets of a complex fuzzy ideal form ideals of the underlying BCK-algebra, and conversely obtained a characterization theorem using level ideals. We also investigated the behavior of intersections of complex fuzzy ideals and studied the structure and ordering properties of level ideals.

These results provide a natural extension of classical fuzzy ideal theory into the complex fuzzy setting and demonstrate that the min-complex t-norm framework is suitable for studying algebraic structures equipped with complex-valued uncertainty. We expect that the methods developed in this work can be further applied to other algebraic systems such as BCI-algebras, MV-algebras, BL-algebras, and related logical structures. Future research may include the study of complex fuzzy implicative ideals, commutative ideals, and other generalized complex fuzzy algebraic concepts under various complex t-norms.

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