

COINCIDENCE POINTS OF MAPPINGS ON RELATIONAL METRIC SPACES WITH APPLICATIONS

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ABSTRACT. In this paper, we prove the existence of coincidence points under non-linear contraction on the settings of metric spaces endowed with an arbitrary binary relation. A suitable non-trivial example is provided to validate the claims of such findings. In addition, we also present an application for the utility of such newly obtained coincidence point results by solving the system of nonlinear fractional differential equations of Caputo type.

1. INTRODUCTION

A generalization of the classical Banach contraction principle [8] to the theme of partial order metric settings can be traced back to Turinici [26, 27, 20, 21]. Recently, a large number of order-theoretic metrical fixed point results have been established for order-preserving contraction mappings. After the publication of Ran and Reuring's [17] theorem and Nieto-López's [14] theorem, the domain of fixed point theory pertained to partially ordered metric spaces has gotten ample attention. There exists considerable theory on this theme, but keeping in view of the necessity of this research presentation, we precisely refer to [2, 4, 18, 5, 24].

Among all these generalizations, we must revisit Samet and Turinici [18] and Alam and Imdad [4] wherein they made use of relational variants of metrical analog of continuity, contractions and completeness etc. to acquire some interesting generalizations of the fixed point results. Further, Alam and Imdad [3] extended the above settings for pair of monotone mappings and obtained common and coincidence

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point theorems for nonlinear contractions (see [16, 10, 6, 22, 23]).

In order to understand many diversified physical problems modeled on functional analysis, the above theoretical study has been very useful. In recent times, there has been keen interest in adopting the generalizations of the Banach contraction principle for the same. The main goal is to present coincidence point results for a pair of mappings on the settings of metric spaces equipped with locally $\mathcal{B}$ -transitive binary relation. Also in view of more comprehensive relational metric settings and the large class of contractions, the obtained coincidence point results generalize, modify and enrich some prominent recent coincidence point theorems in the existing literature. Moreover, to explore the possibility of their applications, we furnish some non-trivial examples and solve the system of nonlinear fractional differential equation of Caputo type and the system of Fredholm integral equations.

2. PRELIMINARIES

In this section, we discuss relevant necessary background for this presentation. Throughout this presentation, $\mathcal{R}$ denotes a non-empty binary relation, $\mathbb{N}$ represents the set of natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 2.1. Let $\mathcal{M}$ be a set and $\mathcal{R} \subseteq \mathcal{M} \times \mathcal{M}$. Then

- (a) we say “ u relates v under $\mathcal{R}$ ” if $(u, v) \in \mathcal{R}$,
- (b) u and v are said to be $\mathcal{R}$ -comparative, if either $(u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}$, and denoted by $[u, v] \in \mathcal{R}$,
- (c) the inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = (u, v) \in \mathcal{M} \times \mathcal{M} : (v, u) \in \mathcal{R}$.
- (d) the symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$, that is, $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$. In fact, $\mathcal{R}^s$ is the smallest symmetric relation on $\mathcal{M}$ containing $\mathcal{R}$.

Definition 2.2. [4, 6]. Let $\mathcal{M}$ be a set equipped with a binary relation $\mathcal{R}$. Let $\mathcal{A}$ and $\mathcal{B}$ be self-mappings on $\mathcal{M}$. Then for $u, v \in \mathcal{M}$,

- (a) $\mathcal{R}$ is called $\mathcal{A}$ -closed if $(u, v) \in \mathcal{R} \Rightarrow (\mathcal{A}u, \mathcal{A}v) \in \mathcal{R}$,
- (b) $\mathcal{R}$ is called $(\mathcal{A}, \mathcal{B})$ -closed if, $(\mathcal{B}u, \mathcal{B}v) \in \mathcal{R} \Rightarrow (\mathcal{A}u, \mathcal{A}v) \in \mathcal{R}$,
- (c) $\mathcal{R}$ is called $(\mathcal{A}, \mathcal{B})$ -closed if $\mathcal{R}^s$ is $(\mathcal{A}, \mathcal{B})$ -closed.

Definition 2.3. [4]. Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{R}$ be a binary relation on $\mathcal{M}$. Then a sequence $\{u_n\} \subset \mathcal{M}$ is said to be

- (a) $\mathcal{R}$ -preserving if $(u_n, u_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$,
- (b) convergent to a point $u \in \mathcal{M}$, i.e. $\lim_{n \rightarrow \infty} d(u_n, u) = 0$,
- (c) Cauchy if $\lim_{n, m \rightarrow \infty} d(u_n, u_m) = 0$.

Definition 2.4. [5]. Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{R}$ be a binary relation on $\mathcal{M}$. Then $(\mathcal{M}, d, \mathcal{R})$ is called an $\mathcal{R}$ -complete relational metric space if every $\mathcal{R}$ -preserving Cauchy sequence converges to a point in $\mathcal{M}$.

Noticeably, completeness of metric spaces implies $\mathcal{R}$ -completeness but not conversely. In fact, in the light of the universal binary relation the notion of $\mathcal{R}$ -completeness coincides with that of usual completeness.

Definition 2.5. [4, 6]. Let $\mathcal{A}$ and $\mathcal{B}$ be self-mappings of a relational metric space $(\mathcal{M}, d, \mathcal{R})$ equipped with a binary relation $\mathcal{R}$ on $\mathcal{M}$.

- (a) $\mathcal{A}$ is $\mathcal{R}$ -continuous at u if for any $\mathcal{R}$ -preserving sequence $\{u_n\}$ so that $u_n \xrightarrow{d} u$, we have $\mathcal{A}u_n \xrightarrow{d} \mathcal{A}u$. In addition, $\mathcal{A}$ is $\mathcal{R}$ -continuous if it is $\mathcal{R}$ -continuous at each point of $\mathcal{M}$.
- (b) $\mathcal{A}$ is $(\mathcal{B}, \mathcal{R})$ -continuous at u if for any sequence $\{u_n\}$ so that $\mathcal{B}u_n$ is $\mathcal{R}$ -preserving and $\mathcal{B}u_n \xrightarrow{d} \mathcal{B}u$, we have $\mathcal{A}u_n \xrightarrow{d} \mathcal{A}u$. In addition, $\mathcal{A}$ is $(\mathcal{B}, \mathcal{R})$ -continuous if it is $(\mathcal{B}, \mathcal{R})$ -continuous at each point of $\mathcal{M}$.

Remark 2.6. Noticeably, continuity in metric spaces implies $\mathcal{R}$ -continuity and $\mathcal{B}$ -continuity implies $(\mathcal{B}, \mathcal{R})$ -continuity under any non-empty binary relation $\mathcal{R}$. In fact, in each case converse is not true and in the light of the universal relation these notions coincides with their corresponding usual notions respectively.

The notion of d -self-closed property of a relational metric space was defined by Turinici [27, 21] as follows:

Definition 2.7. Let $(\mathcal{M}, d, \mathcal{R})$ be a relational metric space. Then

- (a) $\mathcal{R}$ is called d -self-closed if for any $\mathcal{R}$ -preserving sequence $\{u_n\} \subset \mathcal{M}$, so that $u_n \xrightarrow{d} u$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}$,
- (b) $\mathcal{R}$ is called $(\mathcal{B}, d)$ -self-closed if for any $\mathcal{R}$ -preserving sequence $\{u_n\} \subset \mathcal{M}$, so that $u_n \xrightarrow{d} u$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[\mathcal{B}u_{n_k}, \mathcal{B}u] \in \mathcal{R}$.

Inspired by Turinici [27, 21], Alam and Imdad [5] introduced the notion of local transitivity as follows:

Definition 2.8. [5]. Let $\mathcal{M}$ be a set equipped with a binary relation $\mathcal{R}$. Let $\mathcal{A}$ be a self-mapping on $\mathcal{M}$. Then for $u, v, w \in \mathcal{M}$,

- (a) $\mathcal{R}$ is called $\mathcal{A}$ -transitive if, $(\mathcal{A}u, \mathcal{A}v), (\mathcal{A}v, \mathcal{A}w) \in \mathcal{R} \Rightarrow (\mathcal{A}u, \mathcal{A}w) \in \mathcal{R}$;
- (b) $\mathcal{R}$ is called “locally $\mathcal{A}$ -transitive” if for each denumerable (effectively) subset E of $\mathcal{A}(\mathcal{M})$, the binary relation $\mathcal{R}|_E$ is transitive. Here $\mathcal{R}|_E$ is the restriction of $\mathcal{R}$ to E .

Definition 2.9. [5]. Let $\mathcal{M}$ be a non-empty set and $\mathcal{R}$ be a binary relation on $\mathcal{M}$. A subset E of $\mathcal{M}$ is called $\mathcal{R}$ -connected if for each pair $u, v \in E$, there exists a path in $\mathcal{R}$ from u to v .

Definition 2.10. [13]. Let $\mathcal{M}$ be a non-empty set and $\mathcal{R}$ be a binary relation on $\mathcal{M}$. For $u, v \in \mathcal{M}$, a path of length k ($k \in \mathbb{N}$), in $\mathcal{R}$ from u to v is a finite sequence $\{w_0, w_1, w_2, \dots, w_k\} \subset \mathcal{M}$ satisfying the subsequent assumptions:

- (a) $w_0 = u$ and $w_k = v$,
- (b) $(w_i, w_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Noticeably, a path of length k has $k+1$ elements of $\mathcal{M}$, though they are not distinct.

Definition 2.11. [12]. Let $\mathcal{A}$ and $\mathcal{B}$ be self-mappings defined on a non-empty set $\mathcal{M}$. Then

- (a) $u \in \mathcal{M}$ is a coincidence point of $\mathcal{A}$ and $\mathcal{B}$ if $\mathcal{B}u = \mathcal{A}u$,
- (b) if $w \in \mathcal{M}$ is any point so that $w = \mathcal{A}u = \mathcal{B}u$, then w is a point of coincidence of the mappings $\mathcal{A}$ and $\mathcal{B}$.
- (c) the pair $(\mathcal{A}, \mathcal{B})$ is weakly compatible if $\mathcal{A}$ and $\mathcal{B}$ commute at their coincidence points, that is, $\mathcal{B}(\mathcal{A}u) = \mathcal{A}(\mathcal{B}u)$, whenever $\mathcal{A}u = \mathcal{B}u$.

Definition 2.12. [6]. Let $(\mathcal{M}, d)$ be a metric space, $\mathcal{R}$ be a binary relation on $\mathcal{M}$ and $\mathcal{A}$ and $\mathcal{B}$ two self-mappings on $\mathcal{M}$. Then the pair $(\mathcal{A}, \mathcal{B})$ is said to be $\mathcal{R}$ -compatible if for any sequence $\{u_n\} \subset \mathcal{M}$ such that $\mathcal{A}u_n$ and $\mathcal{B}u_n$ are $\mathcal{R}$ -preserving and $\lim_{n \rightarrow \infty} \mathcal{B}u_n = \lim_{n \rightarrow \infty} \mathcal{A}u_n$, we have

$$\lim_{n \rightarrow \infty} d(\mathcal{A}\mathcal{B}u_n, \mathcal{B}\mathcal{A}u_n) = 0.$$

The following family of control functions was essentially indicated by Boyd and Wong [9] but was later used in Jotić [11].

$\Omega = \{\varphi : [0, \infty) \rightarrow [0, \infty) : \varphi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \rightarrow t+} \varphi(r) < t \text{ for each } t > 0\}$.

Lemma 2.13. [3]. *If $\varphi \in \Omega$ and $a_n \in (0, \infty)$ is a sequence such that $a_{n+1} \leq \varphi(a_n)$ for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} a_n = 0$.*

Lemma 2.14. [5, ?]. *Let $(\mathcal{M}, d)$ be a metric space and $\{u_n\}$ be a sequence in $\mathcal{M}$. If $\{u_n\}$ is not Cauchy, then there exist $\epsilon > 0$ and two subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ such that for all $k \in \mathbb{N}_0$, we have*

$$(i) \ k \leq m_k < n_k,$$

$$(ii) \ d(u_{m_k}, u_{n_k}) > \epsilon,$$

$$(iii) \ d(u_{m_k}, u_{n_{k-1}}) \leq \epsilon.$$

Moreover, suppose that $\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0$, then

$$(iv) \ \lim_{k \rightarrow \infty} d(u_{m_k}, u_{n_k}) = \epsilon,$$

$$(v) \ \lim_{k \rightarrow \infty} d(u_{m_{k+1}}, u_{n_{k+1}}) = \epsilon.$$

Given a binary relation $\mathcal{R}$ and two self-mappings $\mathcal{A}$ and $\mathcal{B}$ defined on a non-empty set $\mathcal{M}$, we use the following notations:

- (i) $C(\mathcal{A}, \mathcal{B}) := \{u \in \mathcal{M} : \mathcal{B}u = \mathcal{A}u\}$, that is, the set of all coincidence points of $\mathcal{A}$ and $\mathcal{B}$,
- (ii) $\overline{C}(\mathcal{A}, \mathcal{B}) := \{\overline{u} \in \mathcal{M} : \overline{u} = \mathcal{B}u = \mathcal{A}u\}$, that is, the set of all points of coincidence of $\mathcal{A}$ and $\mathcal{B}$,
- (iii) $\mathcal{M}(\mathcal{A}, \mathcal{R}) := \{u \in \mathcal{M} : (u, \mathcal{A}u) \in \mathcal{R}\}$,
- (iv) $\mathcal{M}(\mathcal{A}, \mathcal{B}, \mathcal{R}) := \{u \in \mathcal{M} : (\mathcal{B}u, \mathcal{A}u) \in \mathcal{R}\}$.

3. MAIN RESULTS

In this section, at first, we establish coincidence point results under nonlinear contraction. Secondly, we furnish a suitable example and some important remarks.

Theorem 3.1. *Let $(\mathcal{M}, d)$ be an $\mathcal{R}$ -complete metric space endowed with an arbitrary binary relation $\mathcal{R}$ on $\mathcal{M}$. Let $\mathcal{A}$ and $\mathcal{B}$ be self-mappings on $\mathcal{M}$. Suppose that the following conditions hold:*

- (a) $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{B}(\mathcal{M})$;
- (b) $\mathcal{R}$ is $(\mathcal{A}, \mathcal{B})$ -closed and locally $\mathcal{B}$ -transitive;
- (c) $\mathcal{M}(\mathcal{A}, \mathcal{B}, \mathcal{R})$ is non-empty;
- (d) $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{R}$ -continuous mappings;
- (e) there exists $\varphi \in \Omega$ such that

$$d(\mathcal{A}u, \mathcal{A}v) \leq \varphi(d(\mathcal{B}u, \mathcal{B}v)) \text{ for all } u, v \in \mathcal{M} \text{ with } (\mathcal{B}u, \mathcal{B}v) \in \mathcal{R};$$

(f) $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{R}$ -compatible.

Then $\mathcal{A}$ and $\mathcal{B}$ have a coincidence point.

Proof. In the light of assumption (c), let u_0 be an arbitrary element of $\mathcal{M}(\mathcal{A}, \mathcal{B}, \mathcal{R})$, then $(\mathcal{B}u_0, \mathcal{A}u_0) \in \mathcal{R}$. If $\mathcal{B}u_0 = \mathcal{A}u_0$, then u_0 is a coincidence point of $\mathcal{A}$ and $\mathcal{B}$ and hence we are done. Otherwise, if $\mathcal{B}u_0 \neq \mathcal{A}u_0$, then from $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{B}(\mathcal{M})$, we choose $u_1 \in \mathcal{M}$ such that $\mathcal{B}u_1 = \mathcal{A}u_0$. As $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{B}(\mathcal{M})$, we select $u_2 \in \mathcal{M}$ such that $\mathcal{B}u_2 = \mathcal{A}u_1$. Continuing this process, we define a sequence of iterates $\{u_n\} \subset \mathcal{M}$ such that

$$(1) \quad u_n = \mathcal{B}u_{n+1} = \mathcal{A}u_n \text{ for all } n \in \mathbb{N}_0.$$

Now we claim that $\{\mathcal{B}u_n\}$ is an $\mathcal{R}$ -preserving sequence, that is,

$$(2) \quad (\mathcal{B}u_n, \mathcal{B}u_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0.$$

We prove this claim by mathematical induction. Using (3.1) (with $n = 0$) and the fact that $u_0 \in \mathcal{M}(\mathcal{A}, \mathcal{B}, \mathcal{R})$, we have

$$(\mathcal{B}u_0, \mathcal{B}u_1) \in \mathcal{R}.$$

Thus, (3.2) holds for $n = 0$. Suppose that (3.2) holds for $n = r > 0$, that is,

$$(\mathcal{B}u_r, \mathcal{B}u_{r+1}) \in \mathcal{R}.$$

Owing to assumption (b), as $\mathcal{R}$ is $(\mathcal{A}, \mathcal{B})$ -closed, we have

$$(\mathcal{A}u_r, \mathcal{A}u_{r+1}) \in \mathcal{R},$$

which, by using (3.1), gives that

$$(\mathcal{B}u_{r+1}, \mathcal{B}u_{r+2}) \in \mathcal{R},$$

that is, (3.2) is valid for $n = r + 1$ too. Hence, by induction, (3.2) remains valid for all $n \in \mathbb{N}_0$.

In the light of the equations (3.1) and (3.2), we conclude that the sequence $\{\mathcal{A}^n u_0\}$ is also $\mathcal{R}$ -preserving, that is,

$$(\mathcal{A}^n u_0, \mathcal{A}^{n+1} u_0) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0.$$

Thus the sequence $\{\mathcal{B}u_n\}$ is $\mathcal{R}$ -preserving. On applying the contractive condition (e) in view of (3.2), we have, for all $n \in \mathbb{N}_0$,

$$d(\mathcal{B}u_{n+1}, \mathcal{B}u_{n+2}) = d(\mathcal{A}u_n, \mathcal{A}u_{n+1}) \leq \varphi(d(\mathcal{B}u_n, \mathcal{B}u_{n+1})).$$

We denote the above sequence by $\{d_n\} \subset (0, \infty)$, such that

$$d_n = d(\mathcal{B}u_n, \mathcal{B}u_{n+1}),$$

and we have $d_{n+1} \leq \varphi(d_n)$. Hence by Lemma 2.13, we obtain

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} d(\mathcal{B}u_n, \mathcal{B}u_{n+1}) = 0.$$

Next, we shall show that $\{\mathcal{B}u_n\}$ is a Cauchy sequence. On contrary suppose that $\{\mathcal{B}u_n\}$ is not a Cauchy sequence. In the light of Lemma 2.14, for $\epsilon > 0$ and for each positive integer k , there exist two subsequences $\{\mathcal{B}u_{n_k}\}$ and $\{\mathcal{B}u_{m_k}\}$ of $\{\mathcal{B}u_n\}$ such that $n_k > m_k \geq k$, $d(\mathcal{B}u_{m_k}, \mathcal{B}u_{n_k}) \geq \epsilon$, $d(\mathcal{B}u_{m_k}, \mathcal{B}u_{n_k+1}) < \epsilon$, and

$$(3) \quad \lim_{k \rightarrow \infty} d(\mathcal{B}u_{m_k}, \mathcal{B}u_{n_k}) = \lim_{k \rightarrow \infty} d(\mathcal{B}u_{m_k+1}, \mathcal{B}u_{n_k+1}) = \epsilon.$$

Denote $r_k = d(\mathcal{B}u_{m_k}, \mathcal{B}u_{n_k})$. As $\{\mathcal{B}u_n\}$ is an $\mathcal{R}$ -preserving sequence and $\{\mathcal{B}u_n\} \subset \mathcal{B}(\mathcal{M})$, by local $\mathcal{B}$ -transitivity of $\mathcal{R}$, we have $(\mathcal{B}u_{m_k}, \mathcal{B}u_{n_k}) \in \mathcal{R}$. So, applying the contractive condition (e), we have

$$\begin{aligned} d(\mathcal{B}u_{m_k+1}, \mathcal{B}u_{n_k+1}) &= d(\mathcal{A}u_{m_k}, \mathcal{A}u_{n_k}) \\ &\leq \varphi(d(\mathcal{B}u_{m_k}, \mathcal{B}u_{n_k})) \\ &= \varphi(r_k). \end{aligned}$$

So

$$(4) \quad d(\mathcal{B}u_{m_k+1}, \mathcal{B}u_{n_k+1}) \leq \varphi(r_k).$$

By taking the limit superior as $k \rightarrow \infty$ in (3.4), in the light of (3.3) and definition of Ω , we have

$$\begin{aligned} \epsilon = \limsup_{k \rightarrow \infty} d(\mathcal{B}u_{m_k+1}, \mathcal{B}u_{n_k+1}) &\leq \limsup_{k \rightarrow \infty} \varphi(r_k) \\ &= \limsup_{r_k \rightarrow \epsilon^+} \varphi(r_k) < \epsilon, \end{aligned}$$

which is a contradiction. So $\{\mathcal{B}u_n\}$ is a Cauchy sequence. Therefore $\{\mathcal{B}u_n\}$ is an $\mathcal{R}$ -preserving Cauchy sequence. By $\mathcal{R}$ -completeness of $\mathcal{M}$, there exists $w \in \mathcal{M}$ such that

$$(5) \quad \lim_{n \rightarrow \infty} \mathcal{B}u_n = w.$$

Using (3.1) and (3.5), we have

$$(6) \quad \lim_{n \rightarrow \infty} \mathcal{A}u_n = w.$$

Finally we use assumptions (d) and (f) to accomplish the proof. Using (3.2) and (3.5) in the light of assumption (d), we obtain

$$(7) \quad \lim_{n \rightarrow \infty} \mathcal{A}(\mathcal{B}u_n) = \mathcal{A}(\lim_{n \rightarrow \infty} \mathcal{B}u_n) = \mathcal{A}(w),$$

and

$$(8) \quad \lim_{n \rightarrow \infty} \mathcal{B}(\mathcal{B}u_n) = \mathcal{B}(\lim_{n \rightarrow \infty} \mathcal{B}u_n) = \mathcal{B}(w).$$

Also as $\{\mathcal{A}u_n\}$ and $\{\mathcal{B}u_n\}$ are $\mathcal{R}$ -preserving and $\lim_{n \rightarrow \infty} \mathcal{A}u_n = \lim_{n \rightarrow \infty} \mathcal{B}u_n = w$, due to (3.5) and (3.6) using $\mathcal{R}$ -compatibility of $\mathcal{A}$ and $\mathcal{B}$ we have

$$(9) \quad \lim_{n \rightarrow \infty} d(\mathcal{B}\mathcal{A}u_n, \mathcal{A}\mathcal{B}u_n) = 0.$$

Using (3.7), (3.8), (3.9) and continuity of d , we have

$$\begin{aligned} d(\mathcal{B}w, \mathcal{A}w) &= d(\lim_{n \rightarrow \infty} \mathcal{B}\mathcal{A}u_n, \lim_{n \rightarrow \infty} \mathcal{A}\mathcal{B}u_n) \\ &= \lim_{n \rightarrow \infty} d(\mathcal{B}\mathcal{A}u_n, \mathcal{A}\mathcal{B}u_n) \\ &= 0, \end{aligned}$$

so that

$$\mathcal{B}(w) = \mathcal{A}(w).$$

Hence $w \in \mathcal{M}$ is a coincidence point of $\mathcal{A}$ and $\mathcal{B}$. This completes the proof. $\square$

Theorem 3.2. *Let $(\mathcal{M}, d)$ be an $\mathcal{R}$ -complete metric space endowed with an arbitrary binary relation $\mathcal{R}$ on $\mathcal{M}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two self-mappings on $\mathcal{M}$. Suppose that the conditions (a), (b), (c), (e) and (f) of Theorem 3.1 hold and $\mathcal{B}$ is $\mathcal{R}$ -continuous and $\mathcal{R}$ is $(\mathcal{B}, d)$ -self-closed then $\mathcal{A}$ and $\mathcal{B}$ have a coincidence point.*

Proof. The proof of this theorem starts along the lines of the proof of our earlier proved Theorem 3.1 and runs up to the lines where we have shown that $\{\mathcal{B}u_n\}$ is an $\mathcal{R}$ -preserving Cauchy sequence.

Now assume that $\mathcal{R}$ is $(\mathcal{B}, d)$ -self-closed then there exists a subsequence $\{\mathcal{B}u_{n_k}\}$ of $\{\mathcal{B}u_n\}$ such that

$$(10) \quad [\mathcal{B}\mathcal{B}u_{n_k}, \mathcal{B}w] \in \mathcal{R} \text{ for all } k \in \mathbb{N}_0.$$

Since $\{\mathcal{B}u_{n_k}\} \xrightarrow{d} w$, (3.5), (3.6), (3.8) and (3.9) hold for $\{u_{n_k}\}$ instead of $\{u_n\}$.

Using (3.10), in the light of assumption (e), we have

$$d(\mathcal{A}\mathcal{B}u_n, \mathcal{A}w) \leq \varphi(d(\mathcal{B}\mathcal{B}u_n, \mathcal{B}w)) \text{ for all } n \in \mathbb{N}_0.$$

Now we claim that

$$(11) \quad d(\mathcal{A}Bu_n, \mathcal{A}w) \leq d(\mathcal{B}Bu_n, \mathcal{B}w) \text{ for all } n \in \mathbb{N}.$$

In order to verify this assertion, two different possibilities are arising here. We resolve them by partitioning $\mathbb{N}$ such that $\mathbb{N} = \mathbb{N}^0 \cup \mathbb{N}^+$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that,

- (i) $d(\mathcal{B}Bu_n, \mathcal{B}w) = 0$, for all $n \in \mathbb{N}^0$,
- (ii) $d(\mathcal{B}Bu_n, \mathcal{B}w) > 0$, for all $n \in \mathbb{N}^+$.

In Case (i), $d(\mathcal{B}Bu_n, \mathcal{B}w) = 0$, that is, $\mathcal{B}(Bu_n) = \mathcal{B}w$, making the use of $(\mathcal{A}, \mathcal{B})$ -closedness of $\mathcal{R}$ we have $\mathcal{A}(Bu_n) = \mathcal{A}w$. Thus (3.11) is valid for all $n \in \mathbb{N}^0$. In Case (ii), owing to definition of Ω we have $d(\mathcal{A}Bu_n, \mathcal{A}w) \leq \varphi(d(\mathcal{B}Bu_n, \mathcal{B}w)) < d(\mathcal{B}Bu_n, \mathcal{B}w)$ for all $n \in \mathbb{N}^+$. Finally (3.11) holds for all $n \in \mathbb{N}$.

Using triangular inequality, (3.8), (3.9) and (3.11), we have

$$\begin{aligned} d(\mathcal{B}w, \mathcal{A}w) &\leq d(\mathcal{B}w, \mathcal{B}Au_n) + d(\mathcal{B}Au_n, \mathcal{A}Bu_n) + d(\mathcal{A}Bu_n, \mathcal{A}w) \\ &\leq d(\mathcal{B}w, \mathcal{B}Au_n) + d(\mathcal{B}Au_n, \mathcal{A}Bu_n) + d(\mathcal{B}Bu_n, \mathcal{B}w). \end{aligned}$$

Letting $n \rightarrow \infty$, we have $d(\mathcal{B}w, \mathcal{A}w) \leq 0$. Also we know $d(\mathcal{B}w, \mathcal{A}w) \geq 0$. So we have $\mathcal{B}w = \mathcal{A}w$. Thus $w \in \mathcal{M}$ is a coincidence point of $\mathcal{A}$ and $\mathcal{B}$. This completes the proof. $\square$

In the next example we consider two non-constant discontinuous mappings on the metric spaces endowed with an arbitrary binary relation satisfying nonlinear contractive assumptions.

Example 3.3. Let $\mathcal{M} = [0, 1]$ and d be the standard metric $d(u, v) = |u - v|$ so that $(\mathcal{M}, d)$ is a complete metric space. Define $\mathcal{R} = \{(0, 0)\} \cup \{(\frac{1}{3^n}, \frac{1}{3^n}) : n \in \mathbb{N}\} \subset \mathcal{M} \times \mathcal{M}$ then by routine calculation one may easily verify that $(\mathcal{M}, d, \mathcal{R})$ is an $\mathcal{R}$ -complete relational metric space. Let $\mathcal{P} = \{0\} \cup \left\{ \frac{1}{3^n} : n \in \mathbb{N} \right\} \subset \mathcal{M}$ then define $\mathcal{A}, \mathcal{B} : \mathcal{M} \rightarrow \mathcal{M}$ as

$$\mathcal{A}(u) = \begin{cases} \frac{u}{9}, & \text{if } u \in \mathcal{P}, \\ 0, & \text{if } u \in [0, 1] \setminus \mathcal{P} \end{cases} \quad \text{and} \quad \mathcal{B}(u) = \begin{cases} \frac{u}{3}, & \text{if } u \in \mathcal{P} \\ 0, & \text{if } u \in [0, 1] \setminus \mathcal{P}. \end{cases}$$

In fact, $\mathcal{R}$ is $(\mathcal{A}, \mathcal{B})$ -closed and locally $\mathcal{B}$ -transitive. If we consider $u_0 = 1 \in \mathcal{P}$, we have $(\mathcal{B}u_0, \mathcal{A}u_0) = (\frac{1}{3}, \frac{1}{9})$. Moreover, $\mathcal{A}(\mathcal{M}) = \mathcal{P} \setminus \{1, \frac{1}{3}\} \subset \mathcal{P} \setminus \{1\} = \mathcal{B}(\mathcal{M})$. Now,

define a comparison function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ by $\varphi(t) = \frac{t}{3}$. For $(u, v) \in \mathcal{R}$, we have

$$\varphi(d(\mathcal{B}u, \mathcal{B}v)) = \varphi\left(\left|\frac{u}{3} - \frac{v}{3}\right|\right) = \frac{1}{9}|u - v| = \left|\frac{u}{9} - \frac{v}{9}\right| = d(\mathcal{A}u, \mathcal{A}v).$$

Now, we claim that $\mathcal{R}$ is $(\mathcal{B}, d)$ -self-closed. Let $\{u_n\}$ be any $\mathcal{R}$ -preserving sequence with $u_n \xrightarrow{d} u$, so that $(u_n, u_{n+1}) \in \mathcal{R}$ for all $k \in \mathbb{N}_0$ which implies that $\{u_n\} \subset L = \{0, \} \cup \{\frac{1}{3^n} : n \in \mathbb{N}\}$. As L is closed, we take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} = u$ for all $k \in \mathbb{N}_0$, which implies that $[u_{n_k}, u] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$. This proves our claim.

Finally, it is interesting to note that considered mappings $\mathcal{A}$ and $\mathcal{B}$ are not continuous, although both the mappings are $\mathcal{R}$ -continuous. This ensures the assumption (d) of Theorem 3.1. Thus all the assumptions of Theorems 3.1 and 3.2 are satisfied, hence $(\mathcal{A}, \mathcal{B})$ has a coincidence point (namely $C(\mathcal{A}, \mathcal{B}) = \{0\}$).

We present one more non-trivial example as follows:

Example 3.4. Assume that $\mathcal{M} = [0, 4]$ and ρ is the standard metric defined on $\mathcal{M}$. Let $\mathcal{R} = \{(0, 0), (0, 2), (2, 0), (0, 3), (2, 4), (4, 4)\}$ be an arbitrary binary relation. Then $(\mathcal{M}, \rho, \mathcal{R})$ is a relational metric space. Moreover, it is easy to calculate that $(\mathcal{M}, d, \mathcal{R})$ is an $\mathcal{R}$ -complete relational metric space. Define $\mathcal{A}, \mathcal{B} : \mathcal{M} \rightarrow \mathcal{M}$ as

$$\mathcal{A}(u) = 0, \ u \in [0, 4], \text{ and } \mathcal{B}(u) = \begin{cases} 0, & \text{if } u \in [0, 2], \\ 2, & \text{if } u \in (2, 3], \\ 3, & \text{if } u \in (3, 4]. \end{cases}$$

In fact, $\mathcal{R}$ is $(\mathcal{A}, \mathcal{B})$ -closed and locally $\mathcal{B}$ -transitive. Although $\mathcal{R}$ is not $\mathcal{B}$ -transitive as $(2, 0), (0, 3) \in \mathcal{R}$ but $(2, 3) \notin \mathcal{R}$.

Now, we assert that $\mathcal{R}$ is $(\mathcal{B}, \rho)$ -self-closed. Let $\{\mathcal{B}u_n\}$ be any $\mathcal{R}$ -preserving convergent sequence with the limit $\mathcal{B}u$, so that $(\mathcal{B}u_{n_k}, \mathcal{B}u) \in \mathcal{R}$, $k \in \mathbb{N}_0$ which implies that $\{\mathcal{B}u_n\} \subset \{0, 2, 3\}$. So $[\mathcal{B}u_{n_k}, \mathcal{B}u] \in \mathcal{R}$, $k \in \mathbb{N}_0$. This validates the assertion.

Finally, one may easily validate the presumed condition (e) of Theorem 3.2. Thus all the assumptions (a)-(f) of Theorems 3.1 and 3.2 are verified, therefore $C(\mathcal{A}, \mathcal{B}) = \{0\}$. Hence $\mathcal{A}$ and $\mathcal{B}$ have a coincidence point.

Remark 3.5. In fact, in Example 3.4 the binary relation $\mathcal{R}$ is nonreflexive, nonsymmetric and nontransitive. So, it is not a partial order, quasi-order, near-order and strict order which substantiate the utility of such generalizations over the corresponding several prominent recent fixed point results on this settings.

Theorem 3.6. *In addition to the hypotheses (a)-(f) of Theorems 3.1 and 3.2, suppose that the following condition holds :*

(u₁) $\mathcal{A}(\mathcal{M})$ is $\mathcal{R}_{|\mathcal{B}(\mathcal{M})}$ -connected.

Then $\mathcal{A}$ and $\mathcal{B}$ have a unique point of coincidence.

Proof. The proof of this theorem starts along the lines of the proof of Theorem 4.5 proved in [6] and runs up to the lines where $\lim_{n \rightarrow \infty} t_n^i = 0$, holds (see [Theorem 4.5]{20}). Thereafter following the idea involved in the proof of Theorem 19 in [24] about nonlinear contraction φ , we conclude that $\mathcal{A}$ and $\mathcal{B}$ have a unique point of coincidence. $\square$

4. APPLICATIONS

In this section, we discuss the existence of a solution for the system of nonlinear fractional differential equations.

The theory of nonlinear fractional differential equations is nowadays a burning topic of research due to the its large applications in many branches of mathematical sciences. The main motivation behind this is due to the sharp rise of the subject fractional calculus itself and of having extensive utilizations in various domains of mathematical sciences.

The notion of the fractional derivative was coined by Liouville and Riemann at the end of nineteenth century although the concept of non integer derivative and integral was originated in 1695 by Leibniz. In point of fact, first order differential equations are contemplated as an alternative model to integer differential equation. The notion of Riemann-Liouville derivative was presented by Riemann in 1876. Afterwards several applications of the fractional derivatives on this theme have been discovered and there exist extensive research work on this theme, but keeping requirement of this presentation on first hand, we precisely refer to [15, 19] and references therein.

In the light of above discuss, it is very important and interesting to study fractional models in respect of global optimization. In the starting of twentieth century, yet another notion of fractional derivative was coined by Caputo in the light of Riemann-Liouville fractional integrals, we refer to [1, 7, ?, 28]. In this connection, we present an application for the existence of a solution for the fractional differential equation.

Let us recall some fundamental notions of fractional calculus (see [?, 19]). For a continuous function $g : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of g of order $\beta > 0$ is defined as

$${}^C D^\beta(g(\rho)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s) ds$$

where $n = [\beta] + 1$ with $[\beta]$ denoting the integer part of a positive real number β , and Γ is the gamma function. Consider the nonlinear fractional differential equation

$$(12) \quad {}^C D^\beta(u(\rho)) = \mathcal{K}(\rho, u(\rho))$$

with integral boundary conditions

$$u(0) = 0, \quad u(1) = \int_0^\eta u(s) ds$$

where $1 < \beta \leq 2$, $0 < \eta < 1$, $u \in \mathcal{C}[0, 1]$, and $\mathcal{K} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. It is well known that if f is continuous, then (4.1) is immediately converted into the very familiar integral equation

$$\begin{aligned} u(\rho) = & \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho-s)^{\beta-1} \mathcal{K}(s, u(s)) ds \\ & - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \mathcal{K}(s, u(s)) ds \\ & + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} \mathcal{K}(m, u(m)) dm \right) ds. \end{aligned}$$

Consider the integral equations

$$(13) \quad \begin{cases} u(\rho) = \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho-s)^{\beta-1} \mathcal{K}_1(s, u(s)) ds - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \mathcal{K}_1(s, u(s)) ds \\ \quad + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} \mathcal{K}_1(m, u(m)) dm \right) ds, \\ v(\rho) = \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho-s)^{\beta-1} \mathcal{K}_2(s, v(s)) ds - \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \mathcal{K}_2(s, v(s)) ds \\ \quad + \frac{2\rho}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} \mathcal{K}_2(m, v(m)) dm \right) ds, \end{cases}$$

For $u(\rho), v(\rho) \in \mathcal{M}$, define binary relation as : $\mathcal{R} = \{(u(\rho), v(\rho)) \in \mathcal{M} \times \mathcal{M} : u(\rho) \leq v(\rho) \text{ with } u(\rho)v(\rho) \geq 0, \rho \in [0, 1]\}$ and $d(u, v) = \sup_{\rho \in [a, b]} |u(\rho) - v(\rho)|$. Then $(\mathcal{M}, d, \mathcal{R})$ is an $\mathcal{R}$ -complete relational metric space.

Now, in the light of above discussion, we prove the following existence theorem for the system of fractional differential equation of Caputo type.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two self-mappings of an $\mathcal{R}$ -complete relational metric space $(\mathcal{M}, d, \mathcal{R})$ equipped with binary relation $\mathcal{R}$. Consider the nonlinear fractional differential equations (4.2) and let for each $u(\rho), v(\rho) \in \mathcal{M}$ with $(u(\rho), v(\rho)) \in \mathcal{R}$, the subsequent assumptions hold:*

(1) *for all $s \in [0, 1]$, there exist continuous functions $\mathcal{K}_1, \mathcal{K}_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and a comparison function $\varphi \in \Omega$ such that*

$$|\mathcal{K}_1(s, u(s)) - \mathcal{K}_2(s, v(s))| \leq \frac{5}{\Gamma(\beta + 1)} \varphi(|\mathcal{K}_1(s, u(s)) - \mathcal{K}_2(s, v(s))|),$$

(2) *there exists a sequence $\{u_n(\rho)\}$ in $\mathcal{M}$ satisfying $\lim_{n \rightarrow \infty} \mathcal{A}u_n(\rho) = \lim_{n \rightarrow \infty} \mathcal{B}u_n(\rho) = w(\rho)$,*

$w(\rho) \in \mathcal{M}$, for an $\mathcal{R}$ -preserving sequence $\mathcal{A}u_n(\rho)$ and $\mathcal{B}u_n(\rho)$ so that $\lim_{n \rightarrow \infty} \mathcal{B}(\mathcal{A}u_n(\rho)) = \mathcal{B}w(\rho)$, $\lim_{n \rightarrow \infty} \mathcal{A}(\mathcal{B}u_n(\rho)) = \mathcal{A}w(\rho)$, and $\lim_{n \rightarrow \infty} \mathcal{A}\mathcal{B}u_n(\rho) = \lim_{n \rightarrow \infty} \mathcal{B}\mathcal{A}u_n(\rho)$,

(3) *$\mathcal{A}u(\rho) = \mathcal{B}u(\rho)$ implies $\mathcal{B}(\mathcal{A}u(\rho)) = \mathcal{A}(\mathcal{B}u(\rho))$, $u(\rho) \in \mathcal{M}$.*

Then the problem (4.2) has a solution.

Proof. Define two self-mappings $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{M}$, such that

$$\begin{aligned} (\mathcal{A}u)(\rho) &= \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho - s)^{\beta-1} \mathcal{K}_1(s, u(s)) ds \\ &\quad - \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} \mathcal{K}_1(s, u(s)) ds \\ &\quad + \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s - m)^{\beta-1} \mathcal{K}_1(m, u(m)) dm \right) ds, \\ (\mathcal{B}v)(\rho) &= \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho - s)^{\beta-1} \mathcal{K}_2(s, v(s)) ds \\ &\quad - \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} \mathcal{K}_2(s, v(s)) ds \\ &\quad + \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s - m)^{\beta-1} \mathcal{K}_2(m, v(m)) dm \right) ds, \end{aligned}$$

for all $u(\rho), v(\rho) \in \mathcal{M}$ and $s, \rho \in [0, 1]$.

For $u(\rho), v(\rho) \in \mathcal{M}$, $[\mathcal{B}u(\rho), \mathcal{B}v(\rho)] \in \mathcal{R}$, that is, either $\mathcal{B}u(\rho) \leq \mathcal{B}v(\rho)$ or $\mathcal{B}u(\rho) \geq$

$\mathcal{B}v(\rho)$, together with $\mathcal{B}u(\rho)\mathcal{B}v(\rho) \geq 0$. We consider the first $\mathcal{B}u(\rho) \leq \mathcal{B}v(\rho)$. Then in view of the assumption (b) of Theorem 3.1, we have $(\mathcal{A}u(\rho), \mathcal{A}v(\rho)) \in \mathcal{R}$, that is, $\mathcal{R}$ is $(\mathcal{A}, \mathcal{B})$ -closed.

Now, we claim that $\mathcal{A}$ and $\mathcal{B}$ satisfy the nonlinear contractive assumption (1). For $u(\rho), v(\rho) \in \mathcal{M}$ and $s, \rho \in [0, 1]$, we have

$$\begin{aligned}
& |(\mathcal{A}u)(\rho) - (\mathcal{A}v)(\rho)| \\
= & \left| \left\{ \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho - s)^{\beta-1} \mathcal{K}_1(s, u(s)) ds - \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} \mathcal{K}_1(s, u(s)) ds \right. \right. \\
& + \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s - m)^{\beta-1} \mathcal{K}_1(m, u(m)) dm \right) ds \Big\} \\
& - \left\{ \frac{1}{\Gamma(\beta)} \int_0^\rho (\rho - s)^{\beta-1} \mathcal{K}_2(s, v(s)) ds - \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} \mathcal{K}_2(s, v(s)) ds \right. \\
& + \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s - m)^{\beta-1} \mathcal{K}_2(m, v(m)) dm \right) ds \Big\} \Big| \\
\leq & \frac{1}{\Gamma(\beta)} \int_0^\rho |\rho - s|^{\beta-1} (|\mathcal{K}_1(s, u(s)) - \mathcal{K}_2(s, v(s))|) ds \\
& - \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} (|\mathcal{K}_1(s, u(s)) - \mathcal{K}_2(s, v(s))|) ds \\
& + \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^\eta \left| \int_0^s (s - m)^{\beta-1} (\mathcal{K}_1(m, u(m)) - \mathcal{K}_2(m, v(m))) dm \right| ds \\
\leq & \frac{1}{\Gamma(\beta)} \int_0^\rho |\rho - s|^{\beta-1} \frac{\Gamma(\beta+1)}{5} \left(\sup_{\rho \in [0,1]} |\mathcal{B}u(\rho) - \mathcal{B}v(\rho)| \right) ds \\
& - \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} \frac{\Gamma(\beta+1)}{5} \left(\sup_{\rho \in [0,1]} |\mathcal{B}u(\rho) - \mathcal{B}v(\rho)| \right) ds \\
& + \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^\eta \left| \int_0^s (s - m)^{\beta-1} \frac{\Gamma(\beta+1)}{5} \left(\sup_{\rho \in [0,1]} |\mathcal{B}u(\rho) - \mathcal{B}v(\rho)| \right) dm \right| ds \\
\leq & \frac{\Gamma(\beta+1)}{5} \left(\sup_{\rho \in [0,1]} |\mathcal{B}u(\rho) - \mathcal{B}v(\rho)| \right) \\
& \times \sup_{\rho \in [0,1]} \left(\frac{1}{\Gamma(\beta)} \int_0^\rho |\rho - s|^{\beta-1} ds \right. \\
& \left. - \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} ds + \frac{2\rho}{(2 - \eta^2)\Gamma(\beta)} \int_0^\eta \int_0^s |s - m|^{\beta-1} dm ds \right) \\
\leq & \varphi(d(\mathcal{B}u(\rho), \mathcal{B}v(\rho))).
\end{aligned}$$

This implies that

$$\sup_{\rho \in [0,1]} |(\mathcal{A}u)(\rho) - (\mathcal{B}v)(\rho)| \leq \varphi(d(\mathcal{B}u(\rho), \mathcal{B}v(\rho))),$$

and so

$$d(\mathcal{A}u, \mathcal{A}v) \leq \varphi(d(\mathcal{B}u, \mathcal{B}v)),$$

for all $u, v \in \mathcal{M}$.

It follows that $\mathcal{A}$ and $\mathcal{B}$ satisfy the contractive assumption. Therefore, all the suppositions of Theorem 3.1 are validated. Hence the system of nonlinear fractional differential equation of Caputo type has a solution. This completes the proof. $\square$

Remark 4.2. In the light of Theorem 3.6 above, we also point out the fact that, if we replace condition (3) of Theorem 4.1 by $\mathcal{A}(\mathcal{M})$ is $\mathcal{R}_{|\mathcal{B}(\mathcal{M})}$ -connected. Then $\mathcal{A}$ and $\mathcal{B}$ have a unique point of coincidence, that is, the system of fractional differential equations (4.2) has a solution.

CONCLUSION

In this work, coincidence point results for nonlinear contraction are proved for a pair of mappings on relational metric space in the light of some weaker relation theoretic metrical variants. Further, we present non-trivial example vindicating that the claims are new original. Indeed, we present variant of prominent recent results on the relational metric settings as underlying nonlinear contraction is assumed to employ only on those elements which belongs to an arbitrary binary relation. In addition, to annotate the utility of such newly obtained results, we solve the system of nonlinear fractional differential equation of Caputo type. Thus, these findings supply yet another view on coincidence point results with some new applications.

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